TD 1: Some Irreversible Processes

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1 Basics of Diffusion. We consider the transport of charges, particles and thermal energy. We denote $\mathbf{j}_q$, $\mathbf{j}_n$ and $\mathbf{j}_U$ the associated currents and ρ_q, ρ_n and ρ_U the associated densities.

1. What are the three ways of transports of heat? Describe their physical origins.
2. Remind Ohm's law, Fick's law and Fourier's law. Comment the differences.
3. What phenomenological approach leads to these laws.
4. Give the name, the dimension and an order of magnitude of the three phenomenological coefficients.
5. We introduce the creation rate by unit volume for each quantity: τ_q, τ_n and τ_U. What can be the physical origin of these rates?
6. What physical principle can be used to get a diffusion equation for each of these quantities? Obtain these equations for temperature and particle density.
7. Find the relaxation equation of the charge.
8. What difference is there between these different processes? What are the scale laws or typical scales for these phenomena?
9. How would these equations be modified in the presence of a drift? In what physical context does this happen?
10. What other diffusion phenomenon do you know?

2 Polymer Diffusion. In this exercise we work in the microcanonical ensemble and give a basic model to describe a polymer. We consider a chain of N monomers of length a. Each monomer is linked to the previous monomer and present a random angle of uniform probability. We are interested in the total length $\mathbf{L}$ of the polymer.

1. What hypotheses are necessary to get a uniform distribution?
2. What is the expectancy $\mathbb{E}(\mathbf{L})$? What is the mean squared expectancy $L_2 = \sqrt{\mathbb{E}(\mathbf{L}^2)}$? Comment.
3. We denote $\mathbb{P}(s, \mathbf{l})$ the probability to reach the position $\mathbf{l}$ after sN monomers. Find an equation of induction for $\mathbb{P}$.
4. We go to the continuum limit by taking $a \to 0$ and $N \to +\infty$ with fixed L_2. Find a partial differential equation for $\mathbb{P}$. What do you recognize?
5. Solve this equation. Is this result surprising? *(hint: it's a Gaussian)*
6. Calculate the entropy $\mathcal{S}(\mathbf{L})$
7. Calculate $\frac{\partial \mathcal{S}}{\partial \mathbf{L}}$. Give a physical interpretation of this result.

3 Shock Waves. We consider a compressible perfect fluid of mass density ρ, velocity field $\mathbf{v}$ and pressure field P.

1. Remind the Euler equations for the fluid. Is this set of equations enough to close the system?
2. We suppose that the process is reversible and adiabatic. Deduce an additional equation for the entropy density s per unit mass.
3. We denote ϵ the internal energy density per unit mass. What is the relation between s and ϵ?
4. We denote e the total energy density. Give its expression.
5. Demonstrate that
$$\partial_t e + \nabla \cdot ((e + P)\mathbf{v}) = 0. \tag{1}$$
6. Deduce that when the flow is stationary, $h_{eff} = \epsilon + \frac{1}{2}\mathbf{v}^2 + \frac{P}{\rho}$ is preserved along the lines of current.

7. What is the velocity of sound c_s? Give its expression.

8. We define the Mach number as $Ma = \frac{v}{c_s}$. Discuss what happens when Ma becomes larger than 1. Make figures.

9. In what follows we consider a stationary fluid and follow a line going from $Ma > 1$ to $Ma < 1$. Show that

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}u} = \rho(1 - Ma^2). \tag{2}$$

10. Plot ρu as a function of u and comment.

    In practice the velocity of sound is dependent of u. We denote it $c(u)$ and the Mach number is $\frac{u}{c(u)}$. We denote with a star (like u^*) the properties at $Ma = 1$ and $Ma^* = \frac{u}{c^*}$. We remind that for a polytropic perfect gas $c^2 = \gamma RT$ and its enthalpy is $\epsilon + \frac{P}{\rho} = C_P T = \frac{c^2}{\gamma - 1}$. We now consider such a gas.

11. Give a relation between Ma and Ma^*. Comment the case $Ma \to +\infty$.

12. Give 3 laws of conservation through the interface $Ma = 1$.

    When the line goes across the sound wall the Prandtl law applies: $u_1 u_2 = c^{*2}$. From these laws specified for the polytropic perfect gas one may calculate

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1) \quad \text{and} \quad \frac{T_2}{T_1} = \left(1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1)\right) \frac{2 + (\gamma - 1)Ma_1^2}{(\gamma+1)Ma_1^2}. \tag{3}$$

13. Calculate the entropy shift $s_2 - s_1$. Comment.

——— *Only when you have finished all the exercises* ———

The Wikipedia Moment. JOSEPH FOURIER (1768-1830).

Fourier was born at Auxerre, the son of a tailor. He was orphaned at the age of nine. Fourier was recommended to the Bishop of Auxerre and, through this introduction, he was educated by the Benedictine Order of the Convent of St. Mark. The commissions in the scientific corps of the army were reserved for those of good birth, and being thus ineligible, he accepted a military lectureship on mathematics. He took a prominent part in his own district in promoting the French Revolution, serving on the local Revolutionary Committee. He was imprisoned briefly during the Terror but, in 1795, was appointed to the École Normale and subsequently succeeded Joseph-Louis Lagrange at the École Polytechnique.

Fourier accompanied Napoleon Bonaparte on his Egyptian expedition in 1798, as scientific adviser, and was appointed secretary of the Institut d'Égypte. Cut off from France by the British fleet, he organized the workshops on which the French army had to rely for their munitions of war. He also contributed several mathematical papers to the Egyptian Institute (also called the Cairo Institute) which Napoleon founded at Cairo, with a view of weakening British influence in the East. After the British victories and the capitulation of the French under General Menou in 1801, Fourier returned to France.

In 1801, Napoleon appointed Fourier Prefect of the Department of Isère in Grenoble, where he oversaw road construction and other projects. However, Fourier had previously returned home from the Napoleon expedition to Egypt to resume his academic post as professor at École Polytechnique when Napoleon decided otherwise. Hence being faithful to Napoleon, he took the office of Prefect. It was while at Grenoble that he began to experiment on the propagation of heat. He presented his paper On the Propagation of Heat in Solid Bodies to the Paris Institute on December 21, 1807. He also contributed to the monumental Description de l'Égypte.

In 1822, Fourier succeeded Jean Baptiste Joseph Delambre as Permanent Secretary of the French Academy of Sciences. He died in his bed on 16 May 1830.