TD 4: Entropy of a Ray of Light

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1 Maximal Work from a Gas of Photons: Carnot Cycle. In this exercise we consider a hot source of volume V at temperatureT_1 emitting light to a cold source at temperature T_2. The hot source is a black body then Stephan's law applies: the emitted power is $\mathcal{P} = \frac{cU}{4V} = \sigma T^4$ where U is the internal energy, c the speed of light and σ is Stephan's constant. Moreover, the pressure can be expressed as $P = \frac{U}{3V}$. We are interested in the maximal work which can be extracted and therefore we assume that we have a Carnot machine to do the transfer.

1. Calculate the heat capacity C of the gas of photons. Is it a heat reservoir? We then denote $T(t)$ the temperature of the hot source, T_1 being its initial temperature. The cold source is assumed large enough to be a thermal reservoir.
2. We denote $\dot{Q}_1$ (resp. $\dot{Q}_2$) the heat provided by the hot source (resp. the cold source) at temperature T (resp. T_2) to the system and $\dot{W}$ the work provided to the system. Give the expression of $\dot{W}$ as a function of C, T and T_2.
3. Calculate the total work W which can be extracted from the gas of photons as a function of $U(T_1)$, T_1 and T_2.
4. Calculate the associated efficiency.
5. The surface temperature of the Sun is $T_1 \sim 4000$ K while for the Earth $T_2 \sim 300$ K. Give an order of magnitude of the maximum efficiency for extracted work from from sunlight.

2 Maximal Work from a Gas of Photons: Thermodynamic Potential. We consider a system of physical properties T, P, V, S, U immersed in a reservoir of volume V_0, pressure P_0, temperature T_0 and entropy S_0. We denote $A = U - T_0 S + P_0 V$ the thermodynamic potential.

1. Prove that A is the thermodynamic potential for this problem.

   We go back to the gas of photons and use the same notations than in the previous exercise. We assume the system 2 to be both a gas of photons and a reservoir.
2. Express the entropy S as a function of V and T.
3. Deduce the maximal work which can be extracted from the gas of photons. Notice that this result does not depend on the change of volume ΔV.

3 Entropy of Radiation. In this exercise we consider a ray of light. We work in the micro-canonical ensemble but we do not assume thermodynamic equilibrium. We define $\mathbb{P}(N_1, N_2, \ldots)$ the probability to find N_1 photons in the state 1, N_2 photons in the state 2. We make 2 assumptions on the form of $\mathbb{P}$. We assume the probability distribution to be of the form $\mathbb{P}(N_1, N_2, \ldots) = p_1(N_1)p_2(N_2)\ldots$. Moreover we assume that the probability to find an electron at state j does not depend on the numbers of photons at the state j.

1. What is the physical meaning o theses 2 assumptions.
2. Show that the entropy is
$$S = -k_B \sum_j \sum_{N_j=0}^{\infty} p_j(N_j)\ln(p_j(N_j)). \tag{1}$$
3. Justify that p_j follows a geometric law and give its expression.
4. We denote n_j the average number of photons in the state j. Express p_j as a function of n_j.
5. Deduce an expression of the entropy as a function of the (n_j).

   We consider a surface dA from which the ray of light is emitted in the direction dΩ which make an angle θ with the normal to the surface. We remind that the number of states of frequencies between ν and $\nu + \mathrm{d}\nu$ can be estimated by $2V\frac{\nu^2}{c^3}\mathrm{d}\nu\mathrm{d}\Omega$.
6. Express s_ν as a function of n_ν and ν such that the entropy transported by the ray is
$$S = V \int s_\nu \mathrm{d}\nu\mathrm{d}\Omega. \tag{2}$$

7. Similarly, express u_ν as a function of n_ν and ν such that the internal energy transported by the ray is

$$U = V \int u_\nu \mathrm{d}\nu \mathrm{d}\Omega. \tag{3}$$

8. We consider the ray at frequency centered at ν_0 with width $\Delta\nu$ and directed to the normal to the surface with width $\Delta\Omega$. Express the flux of energy and entropy per unit surface.

9. Define an effective temperature of the ray. At what condition the flow of energy is non-vanishing for $\Delta\Omega\Delta\nu \to 0$. What is the flow of entropy in this case. Comment.

10. We provide an energy flux $\dot{W}$ and a heat flux $\dot{Q}$ to a light which emits a power $\dot{E}$ and an entropy $\dot{S}$. What is the maximal efficiency when the entropy flux is vanishing? When the system is reversible?

——— *Only when you have finished all the exercises* ———

The Wikipedia Moment. JAMES PRESCOTT JOULE (1818-1889).

James Joule was born in 1818, the son of Benjamin Joule (1784–1858), a wealthy brewer, and his wife, Alice Prescott. Joule was tutored as a young man by the famous scientists. He was fascinated by electricity, and he and his brother experimented by giving electric shocks to each other and to the family's servants.

As an adult, Joule managed the brewery. Science was merely a serious hobby. Sometime around 1840, he started to investigate the feasibility of replacing the brewery's steam engines with the newly invented electric motor. Motivated in part by a businessman's desire to quantify the economics of the choice, and in part by his scientific inquisitiveness, he set out to determine which prime mover was more efficient and discovered Joule's law in 1841.

However, Joule's interest diverted from the narrow financial question to that of how much work could be extracted from a given source, leading him to speculate about the convertibility of energy. In 1843 he published results of experiments showing that the heating effect he had quantified in 1841 was due to generation of heat in the conductor and not its transfer from another part of the equipment. This was a direct challenge to the caloric theory which held that heat could neither be created or destroyed. Caloric theory had dominated thinking in the science of heat since introduced by Antoine Lavoisier in 1783. The practical success of Carnot-Clapeyron's caloric theory of the heat engine since 1824 ensured that the young Joule, working outside either academia or the engineering profession, had a difficult road ahead.

Further experiments and measurements with his electric motor led Joule to estimate the mechanical equivalent of heat as 4.1868 joules per calorie of work to raise the temperature of one gram of water by one Kelvin. He announced his results at a meeting of the chemical section of the British Association for the Advancement of Science in Cork in August 1843 and was met by silence.

Joule was proposing a kinetic theory of heat (he believed it to be a form of rotational, rather than translational kinetic energy), and this required a conceptual leap: if heat was a form of molecular motion, why didn't the motion of the molecules gradually die out? Joule's ideas required one to believe that the collisions of molecules were perfectly elastic. Importantly, the very existence of atoms and molecules was not widely accepted for another 50 years.

Although it may be hard today to understand the allure of the caloric theory, at the time it seemed to have some clear advantages. Carnot's successful theory of heat engines had also been based on the caloric assumption, and only later was it proved by Lord Kelvin that Carnot's mathematics were equally valid without assuming a caloric fluid.

Joule carried experiments on heat and energy during his whole career before Lord Kelvin proved that Carnot's mathematics were equally valid without assuming a caloric fluid. He died in 1889.