TD 11: Fokker-Planck and Noise-Induced Transitions

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Properties of the Wiener process. Consider Langevin dynamics of a particle in a viscous fluid,

$$m\ddot{x} = -\gamma \dot{x} + F(t),$$

where $F(t) = \sigma\gamma\eta(t)$ is a random force, where $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(s) \rangle = 2\delta(t-s)$

1. Show that in the overdamped regime the dynamics reduces to

$$\dot{x} = \sigma\eta(t).$$

2. Writing the fluctuations as $\eta(t)dt = dW_t$ and assuming $x(0) = 0$, show that the solution for $x(t)$ in the overdamped regime is

$$x(t) = \sigma W_t$$

W_t is called the Wiener process and is a fundamental concept in the theory of stochastic processes, and the subscript t in W_t represent the argument of the function.

3. Compute $\langle W_t \rangle$ and $\langle W_t W_s \rangle$.

4. Deduce that $\langle \mathrm{d}W_t^2 \rangle = 2\mathrm{d}t$, implying $\mathrm{d}W_t \overset{ms}{=} O(\sqrt{\mathrm{d}t})$.[1] Are the trajectories defined by the Wiener process continuous? Are they differentiable?

Deriving the Fokker-Planck equation from the Kramers-Moyal coefficients. Recall that the Fokker-Planck equation can be written in terms of the Kramers-Moyal coefficients as,

$$\partial_t p(x,t) = -\partial_x[D_1(x,t)p(x,t)] + \partial_x^2[D_2(x,t)p(x,t)],$$

where,

$$D_n(x,t) = \frac{1}{n!}\lim_{\Delta t \to 0}\frac{1}{\Delta t}\langle (X_{t+\Delta t} - x)^n \rangle|_{X_t = x}\,.$$

5. Leverage the properties we derived for the Wiener process to show that for a stochastic process,

$$\mathrm{d}X_t = a(X_t,t)\mathrm{d}t + b(X_t,t)\mathrm{d}W_t,$$

$D_1(x,t)$ and $D_2(x,t)$ are given by,

$$D_1(x,t) = a(x,t)$$
$$D_2(x,t) = b^2(x,t)$$

6. Show that for an overdamped particle in a potential, $V(x)$, with $\gamma = 1$ and state-dependent (multiplicative) noise $\sigma(x)$, the Fokker-Planck equation is given by,

$$\partial_t p(x,t) = \partial_x[\partial_x V(x)p(x,t)] + \partial_x^2[D(x)p(x,t)]$$

where $D = \sigma^2$.

7. Write down the current J associated with the Fokker-Planck equation,

$$\partial_t p(x,t) = \partial_x J.$$

8. Considering that there is no flux through the domain boundaries, show that the steady-state distribution is given by,

$$p_{st}(x) = \exp\left\{\int^x -\frac{\partial_{x'}V(x') + \partial_{x'}D(x')}{D(x')}dx'\right\},$$

up to a normalization.

9. Show that for additive noise $\sigma(x) = \sigma$ the steady-state distribution reduces to a Boltzmann distribution. Sketch $p_{st}(x)$ for a double well potential with two distinct noise levels $\sigma_1 > \sigma_2$.

10. For multiplicative noise $\sigma(x)$, identify an effective potential $V_{\text{eff}}(x)$. The most likely states, i.e. the maxima of $p_{st}(x)$ or minima of $V_{\text{eff}}(x)$ are no longer at the minimum of the deterministic potential.

[1] The symbol $X \overset{ms}{=} Y$ means equal in mean square, i.e. $\overline{(X-Y)^2} = 0$ implies $X = Y$ almost surely, i.e. X and Y differ at most on a set of measure zero.

Noise-induced transition in logistic growth model. We will now look at an example in which the existence of fluctuations in the model parameters leads to noise-induced transitions between qualitatively different macroscopic behaviors of the system.

11. Consider a model of population growth in which the density of a population $x \in [0,1]$ grows with a rate $\lambda > 0$, but competition for resources yields a decay with $-x^2$, such that,
$$\dot{x} = \lambda x - x^2.$$
Identify its fixed points and their stability w.r.t λ.

12. Compute the solution $x(t)$. Is it consistent with the stability analysis? What happens if we add noise to $\dot{x}$?

13. Consider now that instead of adding noise to $\dot{x}$, we add small Gaussian and white fluctuations in the growth rate, $\lambda \to \lambda'(t) = \lambda + \sigma\eta(t)$. Is the resulting noise additive or multiplicative? Write down the resulting Fokker-Planck equation.

14. Compute the steady-state distribution for vanishing current at the boundaries. What are the most likely states depending in the noise level? Identify a critical noise level at which a transition happens, and describe the different phases.

——— *Only when you have finished all the exercises* ———

The Wikipedia Moment. PAUL LANGEVIN (1872-1946).

Langevin was born in Paris, and studied at the École de Physique et Chimie and the École Normale Supérieure. In 1898, he married Emma Jeanne Desfosses, and together they had four children, Jean, André, Madeleine and Hélène. He then went to the University of Cambridge and studied in the Cavendish Laboratory under Sir J. J. Thomson. Langevin returned to the Sorbonne and obtained his PhD from Pierre Curie in 1902. In 1904, he became Professor of Physics at the Collège de France.

Langevin first worked on paramagnetism and diamagnetism, and developed the modern interpretation of this phenomenon in terms of spins of electrons within atoms. Using the statistical physics of Ludwig Boltzmann, he explained in 1905 why the susceptibility depends on the temperature, a phenomenon which had been discovered experimentally by Pierre Curie. The same year, Albert Einstein interpreted the Brownian motion to estimate the number of Avogadro. At that time, the existence of atoms, which were necessary to statistical physics, was still uncertain. Nevertheless, these results convinced the scientific community to adopt the theory of atoms. In 1908, Langevin gave a formal approach to the Brownian motion by introducing the Langevin equation.

In 1910, he reportedly had an affair with the then-widowed Marie Curie; some decades later, their respective grandchildren, grandson Michel Langevin and granddaughter Hélène Langevin-Joliot married each another. His most famous work was in the use of ultrasound using Pierre Curie's piezoelectric effect. During World War I, he began working on the use of these sounds to detect submarines through echo location. However the war was over by the time it was operational. During his career, Paul Langevin also spread the theory of relativity in academic circles in France and created what is now called the twin paradox.

In 1926, he became director of the École de Physique et Chimie (later became École supérieure de physique et de chimie industrielles de la Ville de Paris, ESPCI ParisTech), where he had been educated. He was elected in 1934 to the Académie des sciences. In 1933, he had a son with physicist Eliane Montel (1898-1993), Paul-Gilbert Langevin, who became a renowned musicologist.

He was also noted for being an outspoken opponent of Nazism, and was removed from his post by the Vichy government following the occupation of the country by Nazi Germany. He was later restored to his position in 1944. His daughter, Hélène Solomon-Langevin, was arrested for Resistance activity and survived several concentration camps. She was on the same convoy of female political prisoners as Marie-Claude Vaillant-Couturier and Charlotte Delbo. He died in Paris in 1946, two years after living to see the Liberation of Paris. He is buried near several other prominent French scientists in the Panthéon in Paris.