TD 13: Stochastic Resonance - Solutions

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Stochastic resonance is a counter-intuitive phenomenon in which periodic structures can be amplified through the presence of noise. It occurs in bistable systems with two inputs: a small-amplitude periodic signal and random fluctuations[1]. As a minimal example with a continuous variable, we consider the Langevin equation

$$\dot{X} = -\frac{dU(x,t)}{dx} + \sigma\eta(t) \tag{1}$$

in a time-dependent double-well potential $U(x,t) = U_0\left[\left(\frac{x}{c}\right)^4 - 2\left(\frac{x}{c}\right)^2\right] - \epsilon_0 \frac{x}{c}\cos(\omega_e t)$ undergoing a small modulation on the external frequency ω_e with amplitude $0 < \epsilon_0 \ll 1$, subject to Gaussian white noise $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$. As we will see, given the right noise level, the system will exhibit a resonance effect in which the signal-to-noise ratio will be higher when noise of a certain magnitude is present.

Time-variation of the potential. Let's get some intuition.

1. Draw the potential for $\epsilon_0 = 0$.
2. Now turn the periodic modulation. What happens to the potential as time goes on? When is the modulated potential most different from $\epsilon_0 = 0$ scenario? For small $\epsilon_0 > 0$, draw the modulated potential at the two time points in which it differs the most from the $\epsilon_0 = 0$ potential.
3. Recall Kramers' result for the mean escape rate from a potential well for small σ,

$$\overline{\tau}(x_0) \sim \frac{2\pi}{\sqrt{V''(x_{min})|V''(x_{barrier})|}} \exp\left[\frac{2}{\sigma^2}(V(x_{barrier}) - V(x_{min})).\right]$$

Compare the escape rates r_- out of the left well and r_+ out of the right well, for the three cases considered in problem 1. How do $r_\pm$ depend on time?

Signal power spectrum. In this section we will estimate the power spectrum of a signal with two macroscopic states and periodic forcing. In order to do that, we will first derive the correlation function through an *adiabatic* approximation of the system, and leverage the Wiener-Kinchin theorem to obtain the power spectrum.

4. Denote by $p(x,t)$ the time-dependent solution of the Fokker-Planck equation. Define $n_- = 1 - n_+ = \int_{-\infty}^{x_b} p(x,t)dx$, where x_b is the position of the potential barrier. What is the interpretation for $n_\pm$?
5. When the hopping rate is much slower than the relaxation to the well we are in an *adiabatic limit*, in which case we can approximate the system as effectively a two state system, with two states at $x \approx +c$ or $x \approx -c$. What is the time scale τ on which a trajectory approaches equilibrium in each well? What should ω_e be for the *adiabatic limit* to be a good approximation?
6. In this limit, write a master equation to find

$$\frac{dn_+}{dt} = r_-(t) - [r_+(t) + r_-(t)]n_+(t).$$

7. Solve this equation to show that, with a function $g(t)$ to be determined,

$$n_+(t) = g^{-1}(t)\left[n_+(t_0)g(t_0) + \int_{t_0}^{t} r_-(t')g(t')dt'\right].$$

8. These integrals cannot be done analytically for the exact Kramers rates. Instead, write $r_\pm = f(\Delta \pm \epsilon\cos(\omega_e t))$, $\Delta \propto U_0/\sigma^2$. Expand for small parameter $\eta = \epsilon\cos(\omega_e t)$, $r_\pm = \frac{\alpha_0}{2} \mp \frac{\alpha_1}{2}\epsilon\cos(\omega_e t) + O(\epsilon^2)$. What are the signs of α_0, α_1?

[1] This exercise follows the classic article *Theory of stochastic resonance* by McNamara, B., & Wiesenfeld, K. (1989), published in Physical review A, 39(9), 4854.

9. Deduce that for a trajectory starting at $x(0) = -c$,

$$n_+(t) = \frac{1}{2}\left\{ e^{-\alpha_0(t-t_0)}\left[2\delta(x_0 - c) - 1 - \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t_0 - \phi)\right] + 1 + \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t - \phi)\right\} + \mathcal{O}(\epsilon^2),$$

where $\delta(x_0 - c)$ is Kronecker delta function, and $\phi = \arctan(\omega_e/\alpha_0)$. You may use the indefinite integral $\int e^{ax}\cos(bx)dx = e^{ax}\frac{\cos(bx-\phi)}{\sqrt{a^2+b^2}}$, with $\phi = \arctan(b/a)$. What changes if we include higher orders in ϵ?

10. In the two-state description, which we have adopted, x is either $+c$ or $-c$. Express the auto-correlation function $\langle x(t+\tau)x(t)|x_0, t_0\rangle$ as a sum of four terms involving products of $(\pm c)n_\pm(t+\tau|\pm c, t)$ and $(\pm c)n_\pm(t|x_0, t_0)$, with $\tau > 0$. Use the normalization condition to find

$$\begin{aligned}\langle x(t)x(t+\tau)|x_0, t_0\rangle =& c^2\left\{[2n_+(t+\tau|c,t) + 2n_+(t+\tau|-c,t) - 2]\, n_+(t|x_0,t_0) + \right.\\ & \left. 1 - 2n_+(t+\tau|-c,t)\right\}\end{aligned}$$

11. *(Bonus)* Take the limit $t_0 \to -\infty$ to find that for $\tau > 0$,

$$\begin{aligned}\langle x(t)x(t+\tau)\rangle \equiv& \lim_{t_0\to-\infty}\langle x(t+\tau)x(t)|x_0,t_0\rangle\\ =& c^2 e^{-\alpha_0\tau}\left[1 - \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\cos^2(\omega_e t - \phi)\right] + \frac{c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\left[\cos(\omega_e(2t+\tau) - 2\phi) + \cos(\omega_e\tau)\right].\end{aligned}$$

You may use the addition theorem $2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$.

12. Given that the auto-correlation function depends periodically on t, experimentalists measure the auto-correlation function for varying t and average across $t \in [0, 2\pi\omega_e]$. Show that

$$\begin{aligned}\overline{\langle x(t)x(t+\tau)\rangle} \equiv& \frac{\omega_e}{2\pi}\int_0^{2\pi/\omega_e}\langle x(t)x(t+\tau)\rangle dt\\ =& c^2 e^{-\alpha_0\tau}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\omega_e^2\cos(\omega_e\tau)}{2(\alpha_0^2+\omega_e^2)}\end{aligned}$$

13. Recall the Wiener-Khinchin theorem relating $\langle x(t)x(t+\tau)\rangle$ and the power spectrum. Denote the time-averaged power spectrum spectrum by $\overline{S_x(\Omega)} = \int_{-\infty}^{\infty}\overline{\langle x(t)x(t+\tau)\rangle}e^{-i\Omega\tau}d\tau$ and show that

$$\overline{S_x(\Omega)} = \frac{2c^2\alpha_0}{\alpha_0^2+\Omega^2}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{\pi c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}[\delta(\Omega-\omega_e) + \delta(\Omega+\omega_e)]$$

14. The one-sided time averaged power spectrum $S_x(\Omega) = \overline{S_x(\Omega)} + \overline{S_x(-\Omega)}$ (defined for positive Ω only) is given by,

$$S_x(\Omega) = \frac{4\alpha_0 c^2}{\alpha_0^2+\Omega^2}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\epsilon^2\pi}{(\alpha_0^2+\omega_e^2)}\delta(\Omega-\omega_e).$$

Notice that the response naturally separates into two parts, one that comes from the broadband noise output which is a Lorentzian bump centered at $\Omega = 0$ and another that comes from the sinusoidal forcing and is a delta function at the signal frequency. The noise spectrum is the product of the Lorentzian obtained with no sinusoidal signal ($\epsilon = 0$), and a correction factor that represents the effects of the signal on the noise. Defining the signal-to-noise ratio as the ratio between the power spectrum of the signal and the noise at the input frequency, derive

$$R = \frac{\epsilon^2\alpha_1^2\pi}{4\alpha_0}\left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right]^{-1}$$

For small ϵ, the correction factor becomes unity and the signal to noise ratio is approximately,

$$R \approx \frac{\epsilon^2\alpha_1^2\pi}{4\alpha_0}$$

Back to the double well example. We will now leverage these results to look at the signal-to-noise ratio resulting from different noise magnitudes and observe the stochastic resonance effect.

15. Compute α_0 and α_1 for the double-well potential and derive that the signal-to-noise ratio is approximately

$$R \approx \frac{4\sqrt{2}U_0\epsilon_0^2 e^{-2U_0/\sigma^2}}{c^2\sigma^4}$$

How does this behave with $D = \sigma^2$?

Correction

1. For $\epsilon = 0$, the potential will have minima at $x_{\pm} = \pm c$ and the barrier will be at $x^* = 0$. The minima will be $U(\pm c, 0) = -U_o$ and the barrier $U(0,0) = 0$, as shown in Fig. 1.

2. When we turn on the modulation for small $\epsilon_0 > 0$, the minima of the potential are roughly in the same location $x_{\pm} \sim \pm c$, but the potential height $U(x^*, t)$ changes in time in a periodic fashion according to $U(\pm c, t) \sim -U_0 \mp \epsilon_0 \cos(\omega_e t)$. Thus the most extreme differences from the $\epsilon_0 = 0$ potential are found when $\cos(\omega_e t) = \pm 1$, which occurs at every $t = 2\pi k/\omega_e$ and $t = \pi(2k+1)/\omega_e$, $k \in \mathbb{N}_0$, as shown in Fig. 2.

3. The energy difference between each minimum and the barrier of the potential is,

$$\Delta U(x_{\pm}) = U(0,t) - U(x_{\pm}, t) = U_0 + \epsilon_0 \frac{x_{\pm}}{c} \cos(\omega_e t).$$

The curvature of the potential at $x^* = 0$ is $U''(x^* = 0, t) = -4U_0/c^2$ and at the minima $x_{\pm}$ is $U''(x^{\pm}, t) = 8U_0/c^2$. Putting all the ingredients together we get,

$$\overline{\tau}(x_{\pm}) = \frac{\pi c^2}{2\sqrt{2U_0}} \exp\left\{\frac{2}{\sigma^2}\left[U_0 + \epsilon_0 \frac{x_{\pm}}{c}\cos(\omega_e t)\right]\right\},$$

and so the escape rates from the wells are given by,

$$r_{\pm}(t) = \frac{2\sqrt{2}U_0}{\pi c^2} \exp\left\{-\frac{2}{\sigma^2}\left[U_0 \pm \epsilon_0 \cos(\omega_e t)\right]\right\}.$$

When $\epsilon_0 = 0$, then there is no difference between r_+ and r_-. When we have an oscillatory input, then the escape rates will also oscillate in time. When $t = 0 \Rightarrow r_+ < r_-$ and when $t = \pi/\omega_e \Rightarrow r_+ > r_-$, reflecting the fact that the right well becomes deeper when $t = 0$ and shallower when $t = \pi/\omega_e$.

4. $n_{\pm}$ is the probability of being in the left (-) or right (+) well.

5. Locally, each potential well is nearly quadratic and so locally the dynamics is like that of an overdamped harmonic oscillator with a time scale that corresponds to the inverse of the curvature at the bottom of the wells, $\tau = [U''(x_{\pm})]^{-1} = c^2/(8U_0)$. This means that the adiabatic limit is attained when $\omega_e \ll 8U_0/c^2$.

6. The master equation for the probability of the right well is,

$$\frac{dn_+}{dt} = -n_+ r_+ + n_- r_-,$$

where the first term accounts for the fraction of probability that escapes the well right transition rate r_+ and the second terms accounts for the fraction of probability that enters the right well with a rate r_-. Leveraging the normalization condition $n_+ + n_- = 1$ we can rewrite this as,

$$\frac{dn_+}{dt} = r_- - n_+(r_+ + r_-)$$

7. Rewriting it as

$$\frac{dn_+}{dt} + n_+(r_+ + r_-) = r_-,$$

we can identify an integrating factor $g(t) = \exp\left(\int^t (r_+(t') + r_-(t'))dt'\right)$, such that we can rewrite the equation as,

$$\frac{d}{dt}\left[g(t)n_+(t)\right] = g(t)r_-(t)$$
$$g(t)n_+(t) = g(t_0)n_+(t_0) + \int_{t_0}^t g(t')r_-(t')dt'$$
$$n_+(t) = g(t)^{-1}\left[g(t_0)n_+(t_0) + \int_{t_0}^t g(t')r_-(t')dt'\right]$$

8. Through the Taylor expansion we get $r_{\pm} \sim f(\Delta) \pm f'(\Delta)\epsilon\cos(\omega_e t) \pm \frac{f''(\Delta)}{2}(\epsilon\cos(\omega_e t))^2 + \mathcal{O}(\epsilon^2)$, where $f' = \frac{df}{d\eta}$ and $\eta = \epsilon\cos(\omega_e t)$. Thus $\alpha_0 = 2f(\Delta) > 0$ and $\alpha_1 = -2f'(\Delta) > 0$ because f must be a decreasing function of its argument: its the transition rate which decays with the energy barrier height.

9. To first order in ϵ, using $r_{\pm} = \frac{\alpha_0}{2} \mp \epsilon\frac{\alpha_1}{2}\cos(\omega_e t)$ we can easily get $g(t)$,

$$g(t') = \exp\left\{\int_{t_0}^{t'} r_+(t') + r_-(t')dt'\right\}$$
$$g(t') = \exp\left\{\int_{t_0}^{t'} \alpha_0 dt'\right\} = e^{\alpha_0(t'-t_0)}$$

And thus,

$$\begin{aligned}
\int_{t_0}^{t} g(t')r_-(t')dt' &= \int_{t_0}^{t} e^{\alpha(t'-t_0)}\left(\frac{\alpha_0}{2}+\epsilon\frac{\alpha_1}{2}\cos(\omega_e t')\right)dt' \\
&= \frac{\alpha_0}{2}\left.\frac{e^{\alpha_0(t'-t_0)}}{\alpha_0}\right|_{t_0}^{t} + \frac{\epsilon\alpha_1}{2}e^{-\alpha_0 t_0}\int_{t_0}^{t}\cos(\omega_e t')e^{\alpha t'}dt' \\
&= \frac{e^{\alpha_0(t-t_0)}-1}{2} + \frac{\epsilon\alpha_1}{2}e^{-\alpha_0 t_0}\left.\frac{e^{\alpha_0 t'}\cos(\omega_e t-\phi)}{\sqrt{\alpha_0^2+\omega_e^2}}\right|_{t_0}^{t} \\
\int_{t_0}^{t} g(t')r_-(t')dt' &= \frac{e^{\alpha_0(t-t_0)}-1}{2} + \frac{\alpha_1\epsilon}{2\sqrt{\alpha_0^2+\omega_e^2}}\left(e^{\alpha_0(t-t_0)}\cos(\omega_e t-\phi)-\cos(\omega_e t_0-\phi)\right),
\end{aligned}$$

where we used the indefinite integral $\int e^{ax}\cos(bx)dx = e^{ax}\frac{\cos(bx-\phi)}{\sqrt{a^2+b^2}}$ and defined $\phi = \arctan(\omega_e/\alpha_0)$. Using these results we can finally write

$$n_+(t) = \frac{1}{2}\left\{e^{-\alpha_0(t-t_0)}\left[2n_+(t_0)-1-\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t_0-\phi)\right]+1+\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t-\phi)\right\}+\mathcal{O}(\epsilon^2)$$

$$n_+(t) = \frac{1}{2}\left\{e^{-\alpha_0(t-t_0)}\left[2\delta(x_0-c)-1-\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t_0-\phi)\right]+1+\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t-\phi)\right\}+\mathcal{O}(\epsilon^2),$$

where $\delta(x_0-c)$ comes from the fact that if the particle is initially at $x_0 = -c$, the initial probability of being in the right well is $n_+(t_0) = 0$. Higher order of ϵ will result in higher powers of sinusoidal functions and thus contribute with higher harmonics in n_+, which will result in higher harmonics in the power spectrum, as we will soon see.

10. The autocorrelation function has 4 terms depending on the state at time t and $t+\tau$. To probability to reach $x = \pm c$ at time t from x_0 and time t_0 will be given by $\pm cn_\pm(t|x_0,t_0)$. The joint probability of $x(t)x(t+\tau)$ can then be written as a sum over 4 terms, depending on whether $x(t) = \pm c$ hops to either $x(t+\tau) = c$ or $x(t+\tau) = -c$, thus,

$$\begin{aligned}
\langle x(t)x(t+\tau)|x_0,t_0\rangle =& cn_+(t|x_0,t_0)cn_+(t+\tau|c,t)+cn_+(t|x_0,t_0)(-c)n_-(t+\tau|c,t)+ \\
&(-c)n_-(t|x_0,t_0)cn_+(t+\tau|-c,t)+(-c)n_-(t|x_0,t_0)(-c)n_-(t+\tau|-c,t),
\end{aligned}$$

writing $n_+ = 1-n_-$, we can obtain,

$$\begin{aligned}
\langle x(t)x(t+\tau)|x_0,t_0\rangle =& c^2\left\{\left[2n_+(t+\tau|c,t)+2n_+(t+\tau|-c,t)-2\right]n_+(t|x_0,t_0)+\right. \\
&\left.1-2n_+(t+\tau|-c,t)\right\}
\end{aligned}$$

11. It's easier to compute the correlation function in steps. First, let's compute $A = 2n_+(t+\tau|c,t)+2n_+(t+\tau|-c,t)-2$, which is,

$$\begin{aligned}
A =& e^{-\alpha_0\tau}\left[1-\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+w_e^2}}\cos(\omega_e t-\phi)\right]+1+\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e(t+\tau)-\phi)+ \\
& e^{-\alpha_0\tau}\left[-1-\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+w_e^2}}\cos(\omega_e t-\phi)\right]+1+\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e(t+\tau)-\phi)-2 \\
=& -2\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t-\phi)e^{-\alpha_0\tau}+2\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e(t+\tau)-\phi) \\
A =& 2\frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\left[\cos(\omega_e(t+\tau)-\phi)-\cos(\omega_e t-\phi)e^{-\alpha_0\tau}\right].
\end{aligned}$$

In the limit $t_0\to-\infty$, $n_+(t|x_0,t_0)$ simplifies to,

$$n_+(t|x_0,t_0) = \frac{1}{2}\left(1+\frac{\epsilon\alpha_1\cos(\omega_e t-\phi)}{\sqrt{\alpha_0^2+\omega_e^2}}\right).$$

And thus the first term becomes

$$\begin{aligned}
An_+(t|x_0,t_0) =& \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\left[\cos(\omega_e(t+\tau)-\phi)-\cos(\omega_e t-\phi)e^{-\alpha_0\tau}\right]+ \\
& \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\left[\cos(\omega_e(t+\tau)-\phi)\cos(\omega_e t-\phi)-\cos^2(\omega_e t-\phi)e^{-\alpha_0\tau}\right].
\end{aligned}$$

The second term $B = 1 - 2n_+(t+\tau| - c, t)$ is,

$$\begin{aligned} B &= -e^{-\alpha_0\tau}\left[-1 - \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t - \phi)\right] - \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e(t+\tau)-\phi) \\ &= e^{-\alpha_0\tau} + \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\left[e^{-\alpha_0\tau}\cos(\omega_e t-\phi) - \cos(\omega_e(t+\tau)\phi)\right] \end{aligned}$$

Finally,

$$\begin{aligned} \langle x(t)x(t+\tau)\rangle &= c^2(An_+(t|x_0,t_0)+B) \\ &= c^2\left\{e^{-\alpha_0\tau} + \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\left[\cos(\omega_e(t+\tau)-\phi)\cos(\omega_e t-\phi) - \cos^2(\omega_e t-\phi)e^{-\alpha_0\tau}\right]\right\} \\ &= c^2\left\{e^{-\alpha_0\tau}\left[1 - \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\cos^2(\omega_e t-\phi)\right] + \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\left[\cos(\omega_e(t+\tau)-\phi)\cos(\omega_e t-\phi)\right]\right\} \\ &= c^2e^{-\alpha_0\tau}\left[1 - \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\cos^2(\omega_e t-\phi)\right] + \frac{c^2\alpha_1^2\epsilon^2/2}{\alpha_0^2+\omega_e^2}\left[\cos(\omega_e(2t+\tau)-2\phi)+\cos(\omega_e\tau)\right] \end{aligned}$$

12. We need to compute

$$\begin{aligned} \overline{\langle x(t)x(t+\tau)\rangle} = \frac{\omega_e}{2\pi}\int_0^{2\pi/\omega_w} & c^2e^{-\alpha_0\tau}\left[1 - \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\cos^2(\omega_e t-\phi)\right] + \\ & \frac{c^2\alpha_1^2\epsilon^2}{2\alpha_0^2+\omega_e^2}\left[\cos(\omega_e(2t+\tau)-2\phi)+\cos(\omega_e\tau)\right]dt. \end{aligned}$$

Splitting the computation into 4 terms, and using $\int_0^{2\pi/a} cos^2(ax-b)dx = \frac{\pi}{a}$ and $\int_0^{2\pi/a} cos(2ax+b) = 0$, we get,

$$\begin{aligned} \overline{\langle x(t)x(t+\tau)\rangle} =& \frac{\omega_e}{2\pi}\left\{c^2e^{-\alpha_0\tau}\frac{2\pi}{\omega_e}\right. \\ & - \frac{c^2e^{-\alpha_0\tau}\alpha_1^2\epsilon^2}{(\alpha_0^2+\omega_e^2)}\frac{\pi}{\omega_e} \\ & + 0 \\ & \left. + \frac{c^2\alpha_1^2\epsilon^2}{\alpha_0^2\omega_e^2}\frac{2\pi}{\omega_e}\cos(\omega_e\tau)\right\} \\ \overline{\langle x(t)x(t+\tau)\rangle} =& c^2e^{-\alpha_0\tau}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\epsilon^2\cos(\omega_e\tau)}{2(\alpha_0^2+\omega_e^2)} \end{aligned}$$

13. Using the fact that the Fourier transform of $e^{-k|x|}$ is a Lorentzian function, $\mathcal{F}_x\left[e^{-k|x|}\right] = \frac{2k}{k^2+\Omega^2}$ and that $\mathcal{F}_x\left[\cos(ax)\right] = \pi\left(\delta(\omega_e-\Omega)+\delta(\omega_e+\Omega)\right)$, we get,

$$\begin{aligned} \overline{S_x(\Omega)} &= c^2\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right)\frac{2\alpha_0}{\alpha_0^2+\Omega^2} + \frac{c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\pi\left[\delta(\omega_e-\Omega)+\delta(\omega_e+\Omega)\right] \\ \overline{S_x(\Omega)} &= \frac{2\alpha_0c^2}{\alpha_0^2+\Omega^2}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\epsilon^2\pi}{2(\alpha_0^2+\omega_e^2)}\left[\delta(\omega_e-\Omega)+\delta(\omega_e+\Omega)\right] \end{aligned}$$

14. The signal-to-noise ratio is given by,

$$\begin{aligned} R &= \frac{\frac{\pi c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}}{\frac{2c^2\alpha_0}{\alpha_0^2+\omega_e^2}\left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right]} \\ &= \frac{\pi\epsilon^2\alpha_1^2}{4\alpha_0}\left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right]^{-1} \end{aligned}$$

15. We saw that $\alpha_0 = 2f(\epsilon=0)$ and $\alpha_1 = -2\frac{df}{d\epsilon}(\epsilon=0)$. Also, the escape rate from the wells is given by,

$$r_\pm(t) = \frac{2\sqrt{2}U_0}{\pi c^2}\exp\left\{-\frac{2}{\sigma^2}\left[U_0 \pm \epsilon_0\cos(\omega_e t)\right]\right\}.$$

Identifying $\Delta = U_0/\sigma^2$ and $\eta = \epsilon\cos(\omega_e t) = \frac{\epsilon_0}{\sigma^2}\cos(\omega_e t)$, we can write,

$$\begin{aligned} f(\Delta+\epsilon\cos(\omega_e t)) &= \frac{2\sqrt{2}U_0}{\pi c^2}\exp\left\{-2\left[\Delta+\epsilon\cos(\omega_e t)\right]\right\} \\ f(\Delta+\eta) &= \frac{2\sqrt{2}U_0}{\pi c^2}\exp\left\{-2\left[\Delta+\eta\right]\right\}, \end{aligned}$$

and thus,

$$\alpha_0 = \frac{4\sqrt{2}U_0}{\pi c^2} e^{-2\frac{U_0}{\sigma^2}}$$
$$\alpha_1 = \frac{8\sqrt{2}U_0}{\pi c^2} e^{-2\frac{U_0}{\sigma^2}}$$
$$\epsilon = \frac{\epsilon_0}{\sigma^2}$$

Plugging this into the expression for the signal-to-noise ratio, R, and using $D = \sigma^2$, we get,

$$R \approx \frac{4\sqrt{2}U_0\epsilon_0^2 e^{-2U_0/D}}{c^2 D^2}.$$

For very small $D \ll U_0$, the exponential decays faster than the denominator and so $R \to 0$. For large D, we need to be careful in interpreting the exponential term since in that regime Kramer's result does not hold (the "saddle point" approximation only holds for $U_0/D \gg 1$). Assuming that the Kramer's regime is still applicable, we get an exponential that converges to 1 but a denominator that grows to $+\infty$, thus when D is large, $R \to 0$ also converges to zero. In between these two regimes, there is an optimal value of the noise magnitude, which corresponds to the barrier height $D \sim U_0$, see Fig. 3. In this regime the response of the system is boosted at the characteristic frequency of the input signal, meaning that noise induces a resonance effect that amplifies the input signal, a counter-intuitive fact denominated **stochastic resonance**.

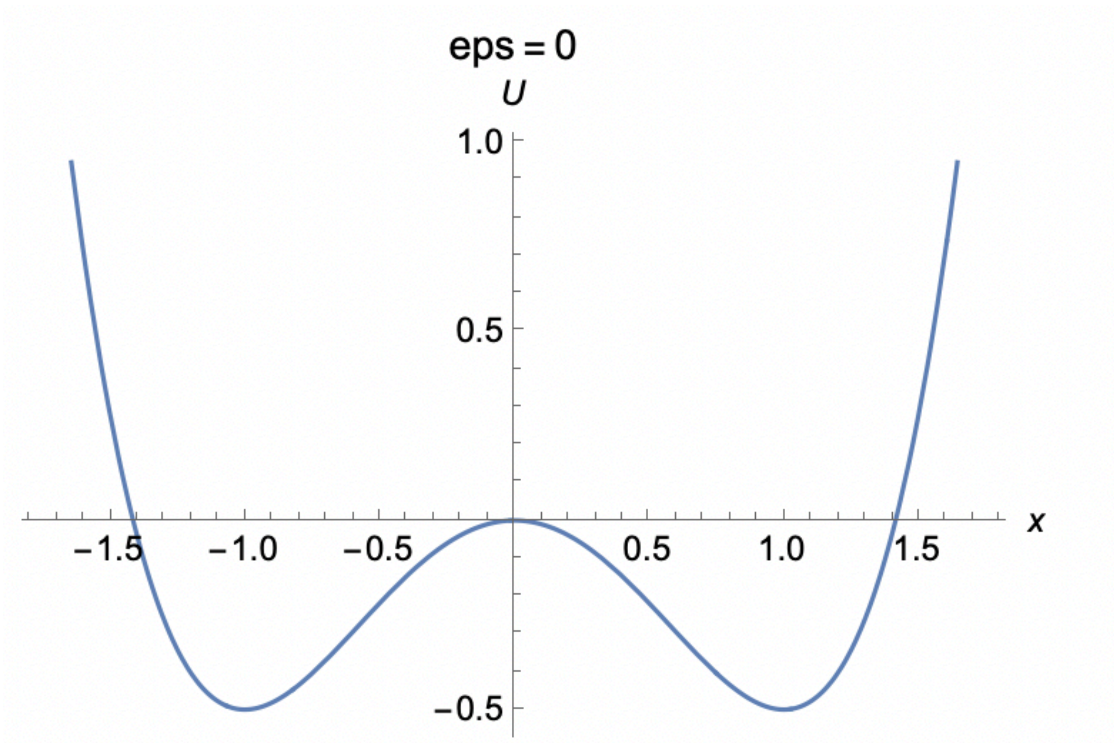

Figure 1: $U(x,t)$ with $U_0 = 1$, $c = 1$, $\omega_e = 1$ and $\epsilon = 0$.

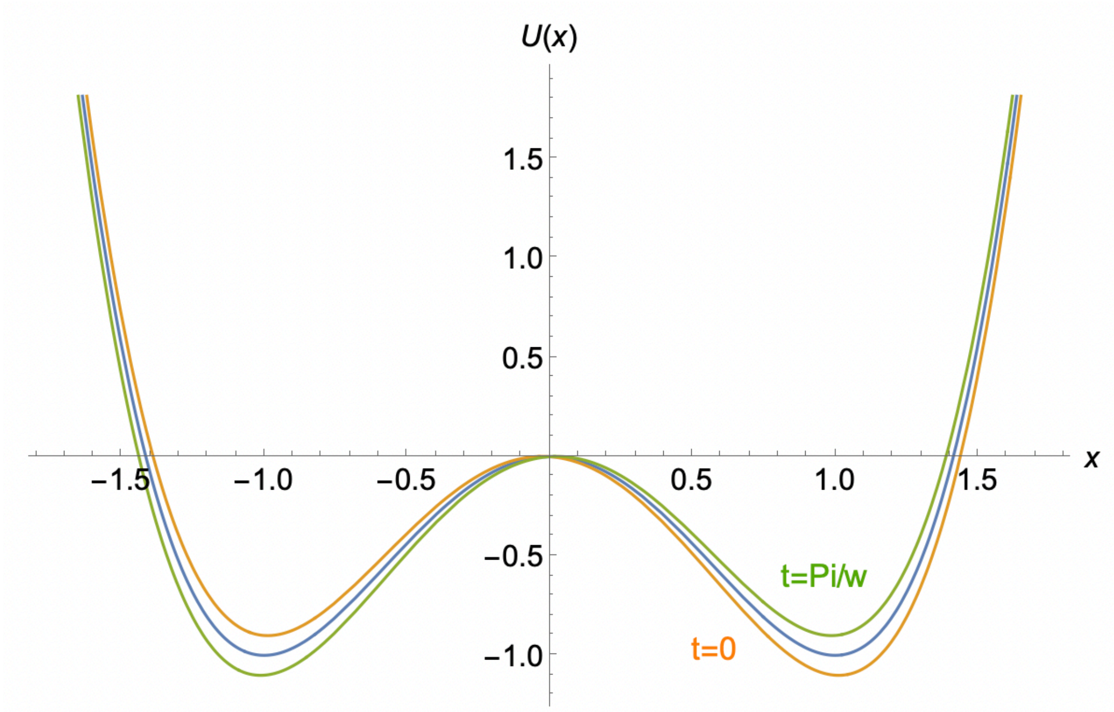

Figure 2: $U(x,t)$ with $U_0 = 1$, $c = 1$, $\omega_e = 1$ and $\epsilon = 0.1$ at $t = 0$ and $t = \pi/\omega_e$.

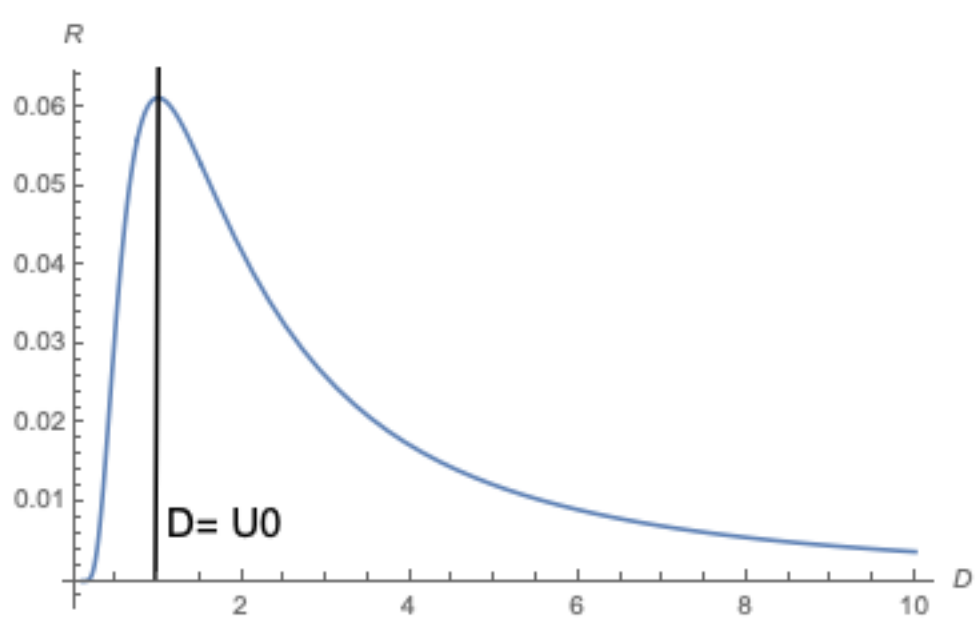

Figure 3: Response R as a function of D. We observe that an intermediate noise level $D \sim U_0$ creates an amplification of the input signal.