TD 13: Stochastic Resonance

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Stochastic resonance is a counter-intuitive phenomenon in which periodic structures can be amplified through the presence of noise. It occurs in bistable systems with two inputs: a small-amplitude periodic signal and random fluctuations[1]. As a minimal example with a continuous variable, we consider the Langevin equation

$$\dot{X} = -\frac{dU(x,t)}{dx} + \sigma\eta(t) \tag{1}$$

in a time-dependent double-well potential $U(x,t) = U_0\left[\left(\frac{x}{c}\right)^4 - 2\left(\frac{x}{c}\right)^2\right] - \epsilon_0\frac{x}{c}\cos(\omega_e t)$ undergoing a small modulation on the external frequency ω_e with amplitude $0 < \epsilon_0 \ll 1$, subject to Gaussian white noise $\langle\eta(t)\rangle = 0$, $\langle\eta(t)\eta(t')\rangle = 2\delta(t-t')$. As we will see, given the right noise level, the system will exhibit a resonance effect in which the signal-to-noise ratio will be higher when noise of a certain magnitude is present.

Time-variation of the potential. Let's get some intuition.

1. Draw the potential for $\epsilon_0 = 0$.
2. Now turn the periodic modulation. What happens to the potential as time goes on? When is the modulated potential most different from $\epsilon_0 = 0$ scenario? For small $\epsilon_0 > 0$, draw the modulated potential at the two time points in which it differs the most from the $\epsilon_0 = 0$ potential.
3. Recall Kramers' result for the mean escape rate from a potential well for small σ,

$$\overline{\tau}(x_0) \sim \frac{2\pi}{\sqrt{V''(x_{min})|V''(x_{barrier})|}}\exp\left[\frac{2}{\sigma^2}(V(x_{barrier}) - V(x_{min})).\right]$$

   Compare the escape rates r_- out of the left well and r_+ out of the right well, for the three cases considered in problem 1. How do $r_\pm$ depend on time?

Signal power spectrum. In this section we will estimate the power spectrum of a signal with two macroscopic states and periodic forcing. In order to do that, we will first derive the correlation function through an *adiabatic* approximation of the system, and leverage the Wiener-Kinchin theorem to obtain the power spectrum.

4. Denote by $p(x,t)$ the time-dependent solution of the Fokker-Planck equation. Define $n_- = 1 - n_+ = \int_{-\infty}^{x_b} p(x,t)dx$, where x_b is the position of the potential barrier. What is the interpretation for $n_\pm$?
5. When the hopping rate is much slower than the relaxation to the well we are in an *adiabatic limit*, in which case we can approximate the system as effectively a two state system, with two states at $x \approx +c$ or $x \approx -c$. What is the time scale τ on which a trajectory approaches equilibrium in each well? What should ω_e be for the *adiabatic limit* to be a good approximation?
6. In this limit, write a master equation to find

$$\frac{dn_+}{dt} = r_-(t) - [r_+(t) + r_-(t)]n_+(t).$$

7. Solve this equation to show that, with a function $g(t)$ to be determined,

$$n_+(t) = g^{-1}(t)\left[n_+(t_0)g(t_0) + \int_{t_0}^{t} r_-(t')g(t')dt'\right].$$

8. These integrals cannot be done analytically for the exact Kramers rates. Instead, write $r_\pm = f(\Delta \pm \epsilon\cos(\omega_e t))$, $\Delta \propto U_0/\sigma^2$. Expand for small parameter $\eta = \epsilon\cos(\omega_e t)$, $r_\pm = \frac{\alpha_0}{2} \mp \frac{\alpha_1}{2}\epsilon\cos(\omega_e t) + O(\epsilon^2)$. What are the signs of α_0, α_1?

[1] This exercise follows the classic article *Theory of stochastic resonance* by McNamara, B., & Wiesenfeld, K. (1989), published in Physical review A, 39(9), 4854.

9. Deduce that for a trajectory starting at $x(0) = -c$,

$$n_+(t) = \frac{1}{2}\left\{ e^{-\alpha_0(t-t_0)}\left[2\delta(x_0-c) - 1 - \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t_0 - \phi)\right] + 1 + \frac{\alpha_1\epsilon}{\sqrt{\alpha_0^2+\omega_e^2}}\cos(\omega_e t - \phi)\right\} + \mathcal{O}(\epsilon^2),$$

where $\delta(x_0 - c)$ is Kronecker delta function, and $\phi = \arctan(\omega_e/\alpha_0)$. You may use the indefinite integral $\int e^{ax}\cos(bx)dx = e^{ax}\frac{\cos(bx-\phi)}{\sqrt{a^2+b^2}}$, with $\phi = \arctan(b/a)$. What changes if we include higher orders in ϵ?

10. In the two-state description, which we have adopted, x is either $+c$ or $-c$. Express the auto-correlation function $\langle x(t+\tau)x(t)|x_0,t_0\rangle$ as a sum of four terms involving products of $(\pm c)n_\pm(t+\tau|\pm c,t)$ and $(\pm c)n_\pm(t|x_0,t_0)$, with $\tau > 0$. Use the normalization condition to find

$$\begin{aligned}\langle x(t)x(t+\tau)|x_0,t_0\rangle =& c^2\left\{\left[2n_+(t+\tau|c,t) + 2n_+(t+\tau|-c,t) - 2\right]n_+(t|x_0,t_0) + \right.\\ &\left. 1 - 2n_+(t+\tau|-c,t)\right\}\end{aligned}$$

11. *(Bonus)* Take the limit $t_0 \to -\infty$ to find that for $\tau > 0$,

$$\begin{aligned}\langle x(t)x(t+\tau)\rangle &\equiv \lim_{t_0\to-\infty}\langle x(t+\tau)x(t)|x_0,t_0\rangle \\ &= c^2 e^{-\alpha_0\tau}\left[1 - \frac{\alpha_1^2\epsilon^2}{\alpha_0^2+\omega_e^2}\cos^2(\omega_e t - \phi)\right] + \frac{c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\left[\cos(\omega_e(2t+\tau) - 2\phi) + \cos(\omega_e\tau)\right].\end{aligned}$$

You may use the addition theorem $2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$.

12. Given that the auto-correlation function depends periodically on t, experimentalists measure the auto-correlation function for varying t and average across $t \in [0, 2\pi\omega_e]$. Show that

$$\begin{aligned}\overline{\langle x(t)x(t+\tau)\rangle} &\equiv \frac{\omega_e}{2\pi}\int_0^{2\pi/\omega_e}\langle x(t)x(t+\tau)\rangle dt \\ &= c^2 e^{-\alpha_0\tau}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\omega_e^2\cos(\omega_e\tau)}{2(\alpha_0^2+\omega_e^2)}\end{aligned}$$

13. Recall the Wiener-Khinchin theorem relating $\overline{\langle x(t)x(t+\tau)\rangle}$ and the power spectrum. Denote the time-averaged power spectrum spectrum by $\overline{S_x(\Omega)} = \int_{-\infty}^{\infty}\overline{\langle x(t)x(t+\tau)\rangle}e^{-i\Omega\tau}d\tau$ and show that

$$\overline{S_x(\Omega)} = \frac{2c^2\alpha_0}{\alpha_0^2+\Omega^2}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{\pi c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\left[\delta(\Omega-\omega_e) + \delta(\Omega+\omega_e)\right]$$

14. The one-sided time averaged power spectrum $S_x(\Omega) = \overline{S_x(\Omega)} + \overline{S_x(-\Omega)}$ (defined for positive Ω only) is given by,

$$S_x(\Omega) = \frac{4\alpha_0 c^2}{\alpha_0^2+\Omega^2}\left(1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right) + \frac{c^2\alpha_1^2\epsilon^2\pi}{(\alpha_0^2+\omega_e^2)}\delta(\Omega-\omega_e).$$

Notice that the response naturally separates into two parts, one that comes from the broadband noise output which is a Lorentzian bump centered at $\Omega = 0$ and another that comes from the sinusoidal forcing and is a delta function at the signal frequency. The noise spectrum is the product of the Lorentzian obtained with no sinusoidal signal ($\epsilon = 0$), and a correction factor that represents the effects of the signal on the noise. Defining the signal-to-noise ratio as the ratio between the power spectrum of the signal and the noise at the input frequency, derive

$$R = \frac{\epsilon^2\alpha_1^2\pi}{4\alpha_0}\left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\omega_e^2)}\right]^{-1}$$

For small ϵ, the correction factor becomes unity and the signal to noise ratio is approximately,

$$R \approx \frac{\epsilon^2\alpha_1^2\pi}{4\alpha_0}$$

Back to the double well example. We will now leverage these results to look at the signal-to-noise ratio resulting from different noise magnitudes and observe the stochastic resonance effect.

15. Compute α_0 and α_1 for the double-well potential and derive that the signal-to-noise ratio is approximately

$$R \approx \frac{4\sqrt{2}U_0\epsilon_0^2 e^{-2U_0/\sigma^2}}{c^2\sigma^4}$$

How does this behave with $D = \sigma^2$?

Only when you have finished all the exercises

The Wikipedia Moment. LARS ONSAGER (1903-1976).

Lars Onsager was born in Kristiania (now Oslo), Norway. His father was a lawyer. After completing secondary school in Oslo, he attended the Norwegian Institute of Technology (NTH) in Trondheim, graduating as a chemical engineer in 1925.

In 1925 he arrived at a correction to the Debye-Hückel theory of electrolytic solutions, to specify Brownian movement of ions in solution, and during 1926 published it. He traveled to Zürich, where Peter Debye was teaching, and confronted Debye, telling him his theory was wrong. He impressed Debye so much that he was invited to become Debye's assistant at the Eidgenössische Technische Hochschule (ETH), where he remained until 1928.

In 1928 he went to the United States to take a faculty position at the Johns Hopkins University in Baltimore, at Brown University in Providence. It quickly became apparent that, while he was a genius at developing theories in physical chemistry, he had little talent for teaching. His research at Brown was concerned mainly with the effects on diffusion of temperature gradients, and produced the Onsager reciprocal relations, a set of equations published in 1929 and, in an expanded form, in 1931, in statistical mechanics whose importance went unrecognized for many years. However, their value became apparent during the decades following World War II, and by 1968 they were considered important enough to gain Onsager that year's Nobel Prize in Chemistry.

In 1933, when the Great Depression limited Brown's ability to support him financially, he joined Yale University, where he remained for most of the rest of his life, retiring in 1972. Just before taking up the position at Yale, Onsager traveled to Austria to visit electrochemist Hans Falkenhagen. He met Falkenhagen's sister-in-law, Margrethe Arledter. They were married on September 7, 1933, and had three sons and a daughter.

During the late 1930s, Onsager researched the dipole theory of dielectrics, making improvements for another topic that had been studied by Peter Debye. During the 1940s, Onsager studied the statistical-mechanical theory of phase transitions in solids, deriving a mathematically elegant theory which was enthusiastically received. In what is widely considered a tour de force of mathematical physics, he obtained the exact solution for the two dimensional Ising model in zero field in 1944.

In 1945, Onsager was naturalized as an American citizen. After World War II, Onsager researched new topics of interest. He proposed a theoretical explanation of the superfluid properties of liquid helium in 1949; two years later the physicist Richard Feynman independently proposed the same theory.

In 1972 Onsager retired from Yale and became emeritus. He died in 1976.