TD 15: Revision - Questions

Baptiste Coquinot & Antonio Costa

January 13, 2022

Consider the dynamics of an overdamped particle influenced by a fluctuating external parameter Λ_t,

$$\dot{x} = F_\Lambda(x) = f(x) + \Lambda_t g(x). \tag{1}$$

Assume that the fluctuations in Λ are fast enough to be modelled as Gaussian white noise, $\Lambda_t = \lambda + \sigma\eta_t$, with $\langle\eta_t\rangle = 0$ and $\langle\eta_t\eta_s\rangle = 2\delta(t-s)$.

1. Identify the deterministic and stochastic components. Is the noise additive or multiplicative?

2. Use the Kramers-Moyal coefficients to deduce the corresponding Fokker-Planck equation,

$$\partial_t\rho(y,t) = -\partial_y\left[F_\lambda(y)\rho(y,t)\right] + \sigma^2\partial_y^2\left[g^2(y)\rho(y,t)\right], \tag{2}$$

   identifying F_λ.

3. Estimate the stationary distribution ρ_s for a system with natural boundary conditions (vanishing probability current), up to a normalization constant N,

$$\rho_s(x) = Ng^{-2}(x)\exp\left[\frac{1}{\sigma^2}\int^x \frac{F_\lambda(u)}{g^2(u)}du\right]. \tag{3}$$

4. Using the previous notation, and $\beta = 1/\sigma^2$, identify an effective potential V_{eff} corresponding to the stationary distribution.

5. Analogous to fixed points in a deterministic dynamical system, the maxima of the stationary distribution correspond to regions of state space where the system lingers more often. Do these maxima correspond to the fixed points of the deterministic system (for which $\sigma = 0$)? Explain while comparing the effects of additive and multiplicative noise.

Population genetics. Consider now the following deterministic model for the fluctuation of the population of an allele in a genetic population, x,

$$\dot{x} = \frac{1}{2} - x + \Lambda x(1-x),\ x \in [0,1], \tag{4}$$

where the rate Λ is related to the selectivity of the allele.

6. Identify the physically relevant fixed point and assess their stability.

7. Imagine now that the environment induces fluctuations in the rate of selection of the allele such that $\Lambda \to \Lambda_t = \lambda + \sigma\eta_t$, where $\langle\eta_t\rangle = 0$, $\langle\eta_t\eta_s\rangle = 2\delta(t-s)$ is Gaussian white noise. Write down the stationary distribution for a vanishing probability current (the final result may contain an unsolved integral).

8. Taking $\lambda = 0$ for simplicity, identify an effective potential and locate the maxima of the stationary distribution.

9. Draw $\rho_s(x)$ for different values of σ, discuss how the noise amplitude results in macroscopically different phases of the system and identify the critical noise level. How is this different from the noiseless case ($\sigma = 0$)?

10. (Bonus) What happens when $|\lambda| > 0$?