

ICFP M1 - PHASE TRANSITIONS – TD n° 4

Correlation Inequalities

Baptiste Coquinot, Guilhem Semerjian

2022 – 2023

We consider a system of N Ising spins, of configurations denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, and of energy

$$H(\underline{\sigma}) = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N J_{i,j} \sigma_i \sigma_j - \sum_{i=1}^N h_i \sigma_i . \quad (1)$$

For a subset of the indices $X \subset \{1, \dots, N\}$ we define $\sigma_X = \prod_{i \in X} \sigma_i$ the product of the spins in this subset. If X is a singleton $\{i\}$ we can use both notations $\sigma_{\{i\}} = \sigma_i$.

We write $\langle \bullet \rangle$ the average on the set of configurations with a Boltzmann weight and an inverse temperature β , i.e. for a function $f(\underline{\sigma})$ we have

$$\langle f(\underline{\sigma}) \rangle = \sum_{\underline{\sigma}} f(\underline{\sigma}) \frac{e^{-\beta H(\underline{\sigma})}}{Z} , \quad Z = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})} . \quad (2)$$

1. What is the physical meaning of the constants $J_{i,j}$ and h_i ? From now on we assume $J_{i,j} \geq 0 \forall i, j$; what does this mean?
2. Assume $h_i \geq 0 \forall i \in \{1, \dots, N\}$. Show that for all subsets $X \subset \{1, \dots, N\}$,

$$\langle \sigma_X \rangle \geq 0 . \quad (3)$$

Hint : Expand the exponential in the numerator of $\langle \sigma_X \rangle$ as a power series.

3. Using the result of the previous question (assuming $h_i \geq 0 \forall i$), show that for all tuples of subsets X, Y ($X \subset \{1, \dots, N\}$, $Y \subset \{1, \dots, N\}$) we have the Griffiths inequality:

$$\langle \sigma_X \sigma_Y \rangle \geq \langle \sigma_X \rangle \langle \sigma_Y \rangle . \quad (4)$$

Hint : One can notice that

$$\langle \sigma_X \sigma_Y \rangle - \langle \sigma_X \rangle \langle \sigma_Y \rangle = \frac{1}{Z^2} \sum_{\underline{\sigma}, \underline{\sigma}'} (\sigma_X \sigma_Y - \sigma_X \sigma_Y') e^{-\beta(H(\underline{\sigma}) + H(\underline{\sigma}'))} \quad (5)$$

and make a change of variable $\tau_i = \sigma_i \sigma_i'$.

4. Let us consider 2 choices $\{h_i^{(1)}, i \in \{1, \dots, N\}\}$ and $\{h_i^{(2)}, i \in \{1, \dots, N\}\}$ of the parameters h_i , and write $\langle \bullet \rangle_1$ and $\langle \bullet \rangle_2$ the averages with respect to these two choices ($J_{i,j}$ being the same in both cases). Using the Fluctuation-Dissipation Theorem and the inequality (??), show that

$$0 \leq h_i^{(1)} \leq h_i^{(2)} \quad \forall i \in \{1, \dots, N\} \quad \Rightarrow \quad \langle \sigma_X \rangle_1 \leq \langle \sigma_X \rangle_2 , \quad (6)$$

whatever the choice of $X \subset \{1, \dots, N\}$.

5. Take the limit of this result for some h_i going to $+\infty$; what is the physical meaning of this limit?
6. How does $\langle \sigma_X \rangle$ vary with respect to the coupling constants $J_{i,j}$ (assuming $h_i \geq 0$)?

7. From this, deduce how $\langle \sigma_X \rangle$ varies with β , for a given value of $J_{i,j} \geq 0$ and $h_i \geq 0$.
8. Application of question ?? : Peierl's argument seen in the lectures shows that in dimension $d = 2$, the spontaneous magnetization is strictly positive for low enough but finite temperatures. Show that this property also holds in $d \geq 3$. *Hint:* For the 3d Ising model consider the case where some coupling constants $J_{i,j}$ are sent to zero such that the model splits in independent parallel 2d Ising models.
9. Application of question ?? : compare the values of T_c for the three regular 2d lattices, namely the triangular, honeycomb and square lattices.