ICFP M1 - Phase Transitions – TD n° 4 – Solution
Correlation Inequalities

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1. $J_{i,j}$ is the spin-spin coupling constant between σ_i and σ_j, h_i the exterior magnetic field acting on the spin σ_i. The condition $J_{i,j} \geq 0$ implies a ferromagnetic behaviour, spins tends to align in the same direction to minimize the energy.

2. The partition function is positive, therefore we only have to show that

$$\sum_{\underline{\sigma}} \sigma_X \prod_{\substack{i,j=1 \\ i \neq j}}^{N} e^{\frac{1}{2}\beta J_{i,j}\sigma_i\sigma_j} \prod_{i=1}^{N} e^{\beta h_i \sigma_i} \geq 0 \ . \tag{1}$$

Expanding the exponential in power series, this expression decompose as a sum of terms proportional (up to positive combinatorial factors) to

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{N} \left(\frac{1}{2}\beta J_{i,j}\right)^{n_{i,j}} \prod_{i=1}^{N} (\beta h_i)^{n_i} \sum_{\underline{\sigma}} \prod_{i \in X} \sigma_i \prod_{\substack{i,j=1 \\ i \neq j}}^{N} (\sigma_i \sigma_j)^{n_{i,j}} \prod_{i=1}^{N} \sigma_i^{n_i} \ , \tag{2}$$

where the $n_{i,j}$ and the n_i are integers ≥ 0. Because $\beta J_{i,j}$ and βh_i are positive, the factor in front of the sum also is. The sum on $\underline{\sigma}$ is also positive: it is equal to 2^N if all spins σ_i are present an even number of time, 0 otherwise.

3. Following the indication,

$$\begin{aligned}
\langle \sigma_X \sigma_Y \rangle - \langle \sigma_X \rangle \langle \sigma_Y \rangle &= \frac{1}{Z^2} \sum_{\underline{\sigma},\underline{\sigma}'} (\sigma_X \sigma_Y - \sigma_X \sigma'_Y) \ e^{\frac{1}{2}\sum\limits_{i,j} \beta J_{i,j}(\sigma_i\sigma_j + \sigma'_i\sigma'_j) + \sum\limits_i \beta h_i (\sigma_i + \sigma'_i)} \\
&= \frac{1}{Z^2} \sum_{\underline{\sigma},\underline{\sigma}'} \sigma_X \sigma_Y (1 - \sigma_Y \sigma'_Y) \ e^{\frac{1}{2}\sum\limits_{i,j} \beta J_{i,j}\sigma_i\sigma_j(1 + \sigma_i\sigma'_i\sigma_j\sigma'_j) + \sum\limits_i \beta h_i \sigma_i (1 + \sigma_i\sigma'_i)} \\
&= \frac{1}{Z^2} \sum_{\underline{\sigma},\underline{\tau}} \sigma_X \sigma_Y (1 - \tau_Y) \ e^{\frac{1}{2}\sum\limits_{i,j} \beta J_{i,j}\sigma_i\sigma_j(1 + \tau_i\tau_j) + \sum\limits_i \beta h_i \sigma_i (1 + \tau_i)} \qquad (3) \\
&= \frac{1}{Z^2} \sum_{\underline{\tau}} (1 - \tau_Y) \sum_{\underline{\sigma}} \sigma_X \sigma_Y \ e^{\frac{1}{2}\sum\limits_{i,j} \beta J_{i,j}(1 + \tau_i\tau_j)\sigma_i\sigma_j + \sum\limits_i \beta h_i (1 + \tau_i)\sigma_i} \ . \qquad (4)
\end{aligned}$$

For $\underline{\tau}$ fixed the sum over $\underline{\sigma}$ is positive according to the previous question: it is indeed proportional to $\langle \sigma_{X\Delta Y} \rangle$ (with $X\Delta Y = (X \cup Y) \setminus (X \cap Y)$ the symmetric difference of the sets X and Y) for a model with couplings $J_{i,j}(1 + \tau_i\tau_j)$ and magnetic fields $h_i(1 + \tau_i)$, which are indeed positive for all $\underline{\tau}$. As $1 - \tau_Y$ is also positive for all $\underline{\tau}$ and Y, all the expression is positive.

4. One can notice that the Fluctuation-Dissipation Theorem gives

$$\frac{\partial}{\partial h_i} \langle \sigma_X \rangle = \beta \left(\langle \sigma_X \sigma_i \rangle - \langle \sigma_X \rangle \langle \sigma_i \rangle \right) \geq 0 \ , \tag{5}$$

the inequality coming from the result of question **??**. Increasing the magnetic field results in an increase of the mean of spin products. By modifying a field once at a time to go from $h_x^{(1)}$ to $h_x^{(2)}$ the means $\langle \sigma_X \rangle$ increase at each step, which yields the inequality of the text.

5. A field h_i equal to $+\infty$ correspond to imposing the value $\sigma_i = +1$ for all configurations with a non-zero probability under the Gibbs-Boltzmann law, all the configurations with $\sigma_i = -1$ having energy equal to $+\infty$, hence zero probability. The inequality in question **??** then shows that fixing some spins to $+1$ increases the magnetization of other spins (which intuitively is quite clear thanks to the ferromagnetic character of the interactions).

6. As in question **??** the Fluctuation-Dissipation Theorem gives

$$\frac{\partial}{\partial J_{i,j}} \langle \sigma_X \rangle = \beta \left(\langle \sigma_X \sigma_i \sigma_j \rangle - \langle \sigma_X \rangle \langle \sigma_i \sigma_j \rangle \right) \geq 0 \ , \tag{6}$$

   i.e. an increase in the coupling constants results in an increase in the average of any product of spins.

7. Since $\langle \sigma_X \rangle$ depends on β only through terms of the form $\beta J_{i,j}$ and βh_i, one can replace the variation of $\langle \sigma_X \rangle$ w.r.t. β by variations w.r.t $J_{i,j}$ and h_i. Then the previous inequalities tell us that $\langle \sigma_X \rangle$ is an increasing function of β.

8. From previous questions, we know that increasing a coupling constant $J_{x,y}$ result in an increase of the mean magnetisation. We use this to first cut the 3d Ising model into independent 2d slices by setting the coupling constant $J_{x,y}$ between the slices to zero. Then we apply Peierl's argument to any 2d slice: there is a finite temperature below which the spontaneous magnetization of the origin $m_{\mathrm{sp}}(T) = \langle \sigma_0 \rangle_{+, L \to \infty}$ is strictly positive in the thermodynamic limit where the positive boundary is sent to infinity. Finally we increase to coupling constants between the slices to retrieve the 3d Ising model and equation (**??**) tells us that this can only lead to an increase of the spontaneous magnetization $m_{\mathrm{sp}}(T)$. More generally the spontaneous magnetization of the Ising model in dimension d increases with d and therefore Peierl's argument can be transferred to any dimension $d \geq 3$.

9. It is possible to go from one planar graph to another by adding or removing bonds, and by stretching the lattices. For example one can go from the square lattice to the triangular lattice by adding one type of diagonal bonds, and from the square to the honeycomb lattice by removing one quarter of the bonds (see the figure below). Additionally we know there has to be a critical temperature in 2d models. Then as in the previous questions, adding bonds increases the spontaneous magnetisation, leading to a possible increase of T_{c}. Hence $T_{\mathrm{c}}^{\text{honey comb}} \leq T_{\mathrm{c}}^{\text{square}} \leq T_{\mathrm{c}}^{\text{triangular}}$.

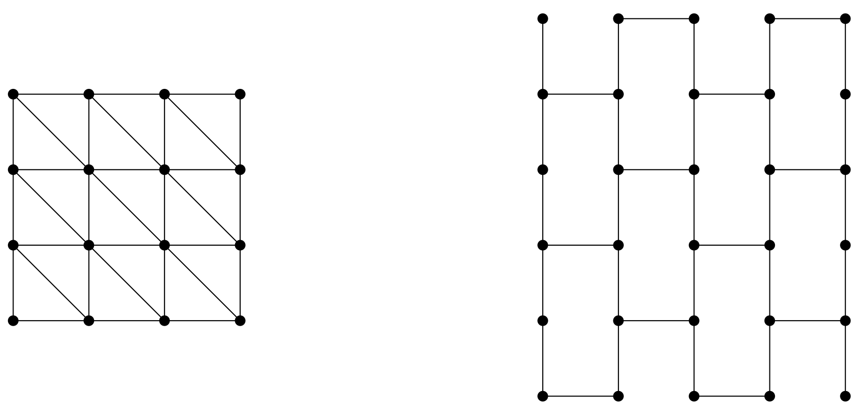

Figure 1: Left: the square lattice with the bonds on one diagonal added, equivalent to the triangular lattice after deformation. Right: the square lattice with one quarter of the bonds removed, that can be stretched into the honeycomb lattice.