ICFP M1 - Phase Transitions – TD n° 5
Real-Space Renormalization Group

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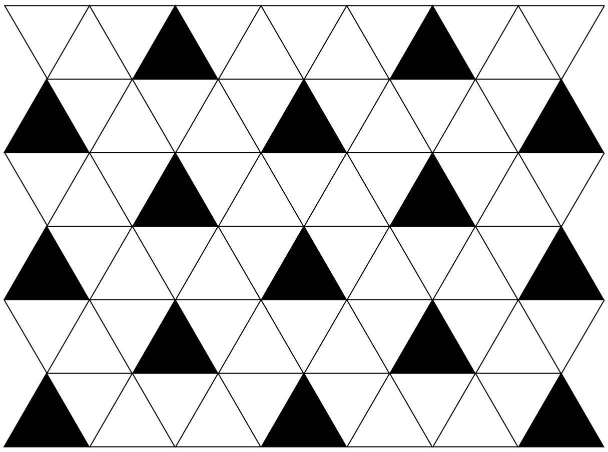
2022 – 2023

We shall consider a system of N Ising spins whose global configuration will be denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N)$, interacting according to the Hamiltonian

$$H(\underline{\sigma}) = -N \, \frac{J_0}{\beta} - \frac{J_1}{\beta} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \frac{J_2}{\beta} \sum_{i=1}^{N} \sigma_i \ , \tag{1}$$

where the $\langle i, j \rangle$ stand for the pairs of nearest neighbors on a bidimensional triangular lattice :



The distance between two nearest neighbors is denoted a.

In this exercice we shall study a transformation of the real-space renormalization group that consist in diminishing the number of degrees of freedom of the system by the introduction of new spins, each of them representing the state of a block of several original spins. The black triangles (blocks) of the figure are indexed by $\alpha = 1, \dots, \widehat{N}$, and we denote $i(\alpha), j(\alpha), k(\alpha)$ the three sites at the vertices of the triangle α. A new Ising spin $\widehat{\sigma}_\alpha$ is placed at the center of each black triangle, and to each configuration $\underline{\sigma}$ of the original spins one associates a configuration $\underline{\widehat{\sigma}} = (\widehat{\sigma}_1, \dots, \widehat{\sigma}_{\widehat{N}})$ of the spins of the new lattice, according to the majority rule inside each block :

$$\widehat{\sigma}_\alpha = \text{sign}\left(\sigma_{i(\alpha)} + \sigma_{j(\alpha)} + \sigma_{k(\alpha)}\right) \ . \tag{2}$$

You will convince yourself that each spin σ_i belongs to one and only one block.

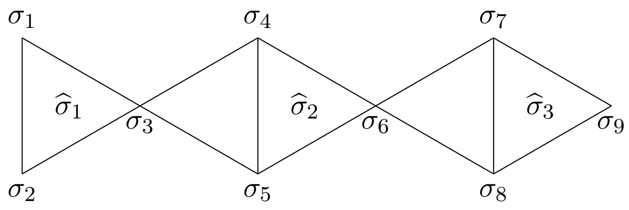
1. What is the distance $\widehat{a}$ between nearest neighbors in the new triangular lattice formed by the blocks α ? We shall denote $b = \widehat{a}/a$ the scale factor of the transformation. What is the number $\widehat{N}$ of spins in the new lattice ? We call $C(\underline{\widehat{\sigma}})$ the set of configurations $\underline{\sigma}$ of the original system that lead to the configuration $\underline{\widehat{\sigma}}$ by the decimation rule (2). What is the cardinality of $C(\underline{\widehat{\sigma}})$?

2. An Hamiltonian $\widehat{H}(\underline{\widehat{\sigma}})$ on the configurations of the new spins is defined according to :

$$\widehat{H}(\underline{\widehat{\sigma}}) = -\frac{1}{\beta} \ln \left[\sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} e^{-\beta H(\underline{\sigma})} \right] \ . \tag{3}$$

Compare the partition functions computed from H and $\widehat{H}$. Interpret the probability of a configuration $\underline{\widehat{\sigma}}$ in the Gibbs-Boltzmann law associated to $\widehat{H}$.

3. In general the exact computation of the Hamiltonian $\widehat{H}$ after the decimation is impossible. To convince oneself of this fact and to understand some properties of the decimation we shall consider, in this question only, a small portion of the triangular lattice made of three blocks :



(a) Write down the most general possible form of $\widehat{H}(\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3)$ as a polynomial in the $\widehat{\sigma}_i$ with terms of degree at most 1 in each variable.

(b) Express formally the coefficients of $\widehat{H}$ in terms of $H(\sigma_1, \ldots, \sigma_9)$, in such a way that they could be computed with a symbolic computation software, MATHEMATICA for instance.

(c) Do you think that some terms of this expansion vanish ?

There is hence no hope to obtain an useful exact form for $\widehat{H}(\underline{\widehat{\sigma}})$. To bypass this difficulty we shall look for an approximation of $\widehat{H}(\underline{\widehat{\sigma}})$ with the same form as $H(\underline{\sigma})$, inspired by the variational method.

4. Consider an arbitrary trial Hamiltonian $H_0(\underline{\sigma})$ (on the initial spins). For each configuration $\underline{\widehat{\sigma}}$ of the decimated spins one introduces an average over the configurations of the initial spins as :

$$\langle \bullet \rangle_{0,\underline{\widehat{\sigma}}} = \frac{1}{Z_0(\underline{\widehat{\sigma}})} \sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} \bullet \, e^{-\beta H_0(\underline{\sigma})} \ , \qquad Z_0(\underline{\widehat{\sigma}}) = \sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} e^{-\beta H_0(\underline{\sigma})} \ . \tag{4}$$

Show that

$$e^{-\beta \widehat{H}(\underline{\widehat{\sigma}})} = Z_0(\underline{\widehat{\sigma}}) \, \langle e^{-\beta(H(\underline{\sigma}) - H_0(\underline{\sigma}))} \rangle_{0,\underline{\widehat{\sigma}}} \ , \tag{5}$$

and deduce from this fact the following upperbound on $\widehat{H}$:

$$\widehat{H}(\underline{\widehat{\sigma}}) \leq -\frac{1}{\beta} \ln Z_0(\underline{\widehat{\sigma}}) + \langle H(\underline{\sigma}) - H_0(\underline{\sigma}) \rangle_{0,\underline{\widehat{\sigma}}} \ . \tag{6}$$

5. We shall take as a trial Hamiltonian

$$H_0(\underline{\sigma}) = -N \, \frac{J_0}{\beta} - \frac{J_1}{\beta} \sum_{\alpha=1}^{\widehat{N}} \sum_{\langle i,j \rangle \in \alpha} \sigma_i \sigma_j \ , \tag{7}$$

where the sum on $\langle i, j \rangle \in \alpha$ involves the three edges inside the triangle α. Compute $Z_0(\underline{\widehat{\sigma}})$. Show that

$$\langle \sigma_i \rangle_{0,\underline{\widehat{\sigma}}} = \frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3 \, e^{-J_1}} \, \widehat{\sigma}_{\alpha(i)} \ , \tag{8}$$

where $\alpha(i)$ is the block to which the site i belongs. What is the value of $\langle \sigma_i \sigma_j \rangle_{0,\underline{\widehat{\sigma}}}$ if i and j belong to two different blocks ?

6. Deduce then the value of $\langle H(\underline{\sigma}) - H_0(\underline{\sigma}) \rangle_{0,\underline{\widehat{\sigma}}}$, and show that

$$\widehat{H}(\underline{\widehat{\sigma}}) \leq -\widehat{N} \, \frac{\widehat{J}_0}{\beta} - \frac{\widehat{J}_1}{\beta} \sum_{\langle \alpha,\beta \rangle} \widehat{\sigma}_\alpha \widehat{\sigma}_\beta - \frac{\widehat{J}_2}{\beta} \sum_{\alpha=1}^{\widehat{N}} \widehat{\sigma}_i \ , \tag{9}$$

where $\langle \alpha, \beta \rangle$ denote the pairs of nearest neighbors blocks in the new lattice, and where the new coupling constants (within this approximation) are given by :

$$\begin{aligned}
\widehat{J}_1 &= 2\; J_1 \left(\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\; e^{-J_1}} \right)^2 \\
\widehat{J}_2 &= 3\; J_2 \left(\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\; e^{-J_1}} \right) \\
\widehat{J}_0 &= 3\; J_0 + \log(e^{3J_1} + 3\; e^{-J_1})
\end{aligned} \tag{10}$$

7. What are the fixed points (J_1^*, J_2^*) of the flow of the coupling constants $(J_1, J_2) \to (\widehat{J}_1, \widehat{J}_2)$? Study their stability, by computing the matrix $M_{ij} = \frac{\partial \widehat{J}_i}{\partial J_j}$.

8. The eigenvalues $\lambda_{1,2}$ of the matrix M, at the non-trivial fixed point, will be written $\lambda_i = b^{y_i}$. Compute $y_{1,2}$ and then the critical exponents using the following relations that will be derived in the lecture next week,

$$\alpha = 2 - \frac{d}{y_1}\ ,\ \beta = \frac{d - y_2}{y_1}\ ,\ \gamma = \frac{2y_2 - d}{y_1}\ ,\ \delta = \frac{y_2}{d - y_2}\ ,\ \nu = \frac{1}{y_1}\ ,\ \eta = d + 2 - 2y_2\ . \tag{11}$$

Compare them to the exactly known values in dimension $d = 2$,

$$\alpha = 0\ ,\ \beta = \frac{1}{8}\ ,\ \gamma = \frac{7}{4}\ ,\ \delta = 15\ ,\ \nu = 1\ ,\ \eta = \frac{1}{4}\ , \tag{12}$$

that follow from $y_1 = 1$, $y_2 = \frac{15}{8}$.