ICFP M1 - Phase Transitions – TD n° 6 – Solution
Scaling Functions and Relationships between Critical Exponents

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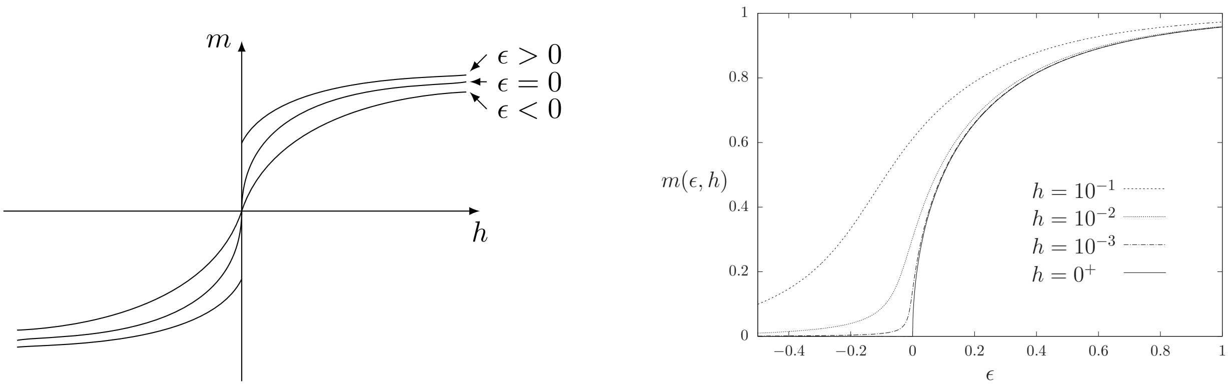
1. The relation of ϵ to the reduced temperature $t = \frac{T-T_c}{T_c}$ is $\epsilon = -\frac{T_c}{T}t$. As $\frac{T_c}{T}$ goes to a strictly positive constant (actually 1) at the critical point one has $\epsilon^a \propto (-t)^a$ for any a, hence the definition of the critical exponents in terms of ϵ or in terms of $-t$ coincide.

2. On the left of the figure below one can see the shape of the magnetization as a function of h for $\epsilon < 0$ (in this high temperature phase it is a continuous function with a finite slope at the origin), for $\epsilon = 0$ (at the critical temperature it is continuous but has an infinite slope at the origin, behaving as $h^{1/\delta}$), and for $\epsilon > 0$ (in the low temperature phase it has a discontinuity, jumping from $-m_{\rm sp}$ to $+m_{\rm sp}$ as h go from $h = 0^-$ to $h = 0^+$). On the right one has the magnetization as a function of ϵ for $h = 0^+$ (this curve vanishes for $\epsilon < 0$, and is equal to the spontaneous magnetization for $\epsilon > 0$), and for a few values of $h > 0$ (the magnetization is then smooth at all temperatures).



1 The mean-field value of the critical exponents

3. We recall that $\tanh(x) = x - x^3/3 + \mathcal{O}(x^5)$ for small x. We have to find the behavior of $m(\epsilon)$, the solution of the implicit equation $m = \tanh((1+\epsilon)m)$, when ϵ is small and positive. In this limit m will be small as well, we can thus expand the right hand side of the equation into

$$m = (1+\epsilon)m - \frac{1}{3}((1+\epsilon)m)^3 + \mathcal{O}(((1+\epsilon)m)^5) = m + \epsilon\, m - \frac{1}{3}m^3 + \mathcal{O}(m^5, m^3\epsilon) \ , \qquad (1)$$

where we also expanded the cubic term at its lowest order in ϵ. Substracting the dominant term m from both sides we obtain

$$m \sim \sqrt{3\epsilon} \ , \qquad (2)$$

which allows to check a posteriori that the terms discarded from equation (1) were indeed negligible with respect to those kept. We thus obtain the critical exponent $\beta = 1/2$.

4. Following the same strategy on the implicit equation $m = \tanh(m+h)$ yields

$$m = m + h - \frac{1}{3}(m+h)^3 + \mathcal{O}((m+h)^5) = m + h - \frac{1}{3}m^3 + \mathcal{O}((m+h)^5, m^2h, mh^2, h^3) \ , \qquad (3)$$

which gives

$$m \sim (3h)^{1/3} \ , \qquad (4)$$

hence the value $\delta = 3$ for this critical exponent.

5. We use $\frac{\mathrm{d}}{\mathrm{d}x}\tanh(x) = 1 - \tanh(x)^2$ to differentiate the implicit equation and find

$$\chi(\epsilon, h) = \frac{(1-m^2)(1+\epsilon)}{1-(1-m^2)(1+\epsilon)} = \frac{1}{1-(1-m^2)(1+\epsilon)} - 1 \ . \tag{5}$$

For $\epsilon < 0$, we use that $m(\epsilon, h = 0^+) = 0$, hence $\chi(\epsilon, 0) = \frac{1}{|\epsilon|} - 1 \propto |\epsilon|^{-1}$ for $\epsilon \to 0^-$.

To study the limit $\epsilon \to 0^+$ we plug in the expression $m \sim \sqrt{3\epsilon}$ from (2). This gives $1-(1-m^2)(1+\epsilon) = 2\epsilon + \mathcal{O}(\epsilon^2)$, hence $\chi(\epsilon, 0) \propto |\epsilon|^{-1}$ for $\epsilon \to 0^+$.

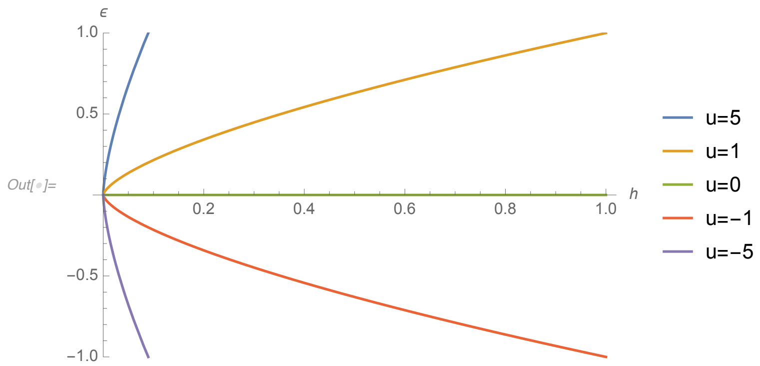
Therefore $\gamma = \gamma' = 1$, the divergence of the susceptibility is governed by the same exponent when the critical point is approached from the low and high temperature phases.

2 Existence of a scaling function in mean-field

6. In the previous questions we have studied the behavior of the magnetization when approaching the critical point $(\epsilon, h) = (0, 0)$ along two different paths: first along the axis $h = 0$, then along the axis $\epsilon = 0$. We want now to determine a path in the (ϵ, h) plane along which both the temperature and the magnetic field have an influence on the magnetization. To do so we have to solve the self-consistent equation $m = \tanh((1+\epsilon)(m+h))$, when ϵ, h and m are small, but of relative orders chosen in an appropriate way for ϵ and h to influence simultaneously m. In equation (1) the terms which played a role were of order $\epsilon\, m$ and m^3, while in equation (3) they were of order h and m^3. For m^3, $\epsilon\, m$ and h to be of the same order we need to have m of order $h^{1/3}$ and ϵ of order $h^{2/3}$.

Alternatively, a more direct way is to exploit the scale laws already obtained. To get a relevant behaviour we indeed need $m \sim h^{1/3} \sim \epsilon^{1/2}$. In particular we recover $\epsilon \sim h^{2/3}$.

7. To have m of order $h^{1/3}$ we take $a = 1/3$, to have ϵ of order $h^{2/3}$ we take $b = 2/3$. We thus approach the critical point along curves in the (ϵ, h) planes parametrized by h, with $\epsilon = u\, h^{2/3}$, various values of u correspond to different curves, as shown on the figure below:



When approaching the origin along a curve for a given value of u the magnetization behaves as $g(u)h^{1/3}$.

8. To find the equation satisfied by $g(u)$, we replace in $m = \tanh((1+\epsilon)(m+h))$ the parametrization of m and ϵ in terms of h,

$$gh^{1/3} = \tanh((1+uh^{2/3})(gh^{1/3}+h)) \ , \tag{6}$$

and expand for small h:

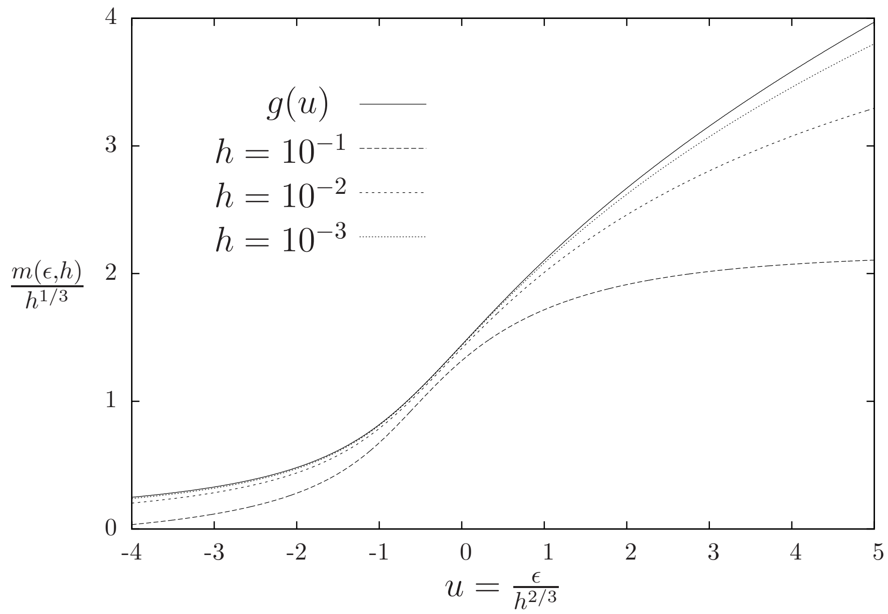
$$\begin{aligned} gh^{1/3} &= (1+uh^{2/3})(gh^{1/3}+h) - \frac{1}{3}(1+uh^{2/3})^3(gh^{1/3}+h)^3 + \mathcal{O}(h^{5/3}) \\ &= g\,h^{1/3} + h\left(ug + 1 - \frac{1}{3}g^3\right) + o(h) \ . \end{aligned}$$

According to our plan we have the three non-trivial terms of the same order, and we find the equation satisfied by the scaling function $g(u)$:

$$0 = 1 + u\, g(u) - \frac{1}{3} g(u)^3 \ . \qquad (7)$$

This is a cubic equation, hence $g(u)$ could be written explicitly as a function of u; the resulting formula is rather cumbersome and not very useful, we shall thus content ourselves with a numerical determination, represented as a solid line on the next figure. To obtain the shape of this function without a computer one can notice that the reciprocal function $u(g)$ is easy to express, $u(g) = \frac{1}{3}g^2 - \frac{1}{g}$. One can study this function, plot it and flip it along the diagonal to have its reciprocal $g(u)$; as $h \geq 0$ implies $m \geq 0$ one can restrict the study to the domain $g \geq 0$, on which $u(g)$ is bijective, there is thus no ambiguity in the inversion between $u(g)$ and $g(u)$. In other words the cubic equation (7) has a unique positive root for all real values of u.

Setting $u = 0$ in the equation (7) one obtains immediately $g(0) = 3^{1/3}$, in agreement with (4). Indeed $u = 0$ corresponds to $\epsilon = 0$, i.e. the approach of the critical point along the h axis. One can easily determine the behaviour of $g(u)$ in $u \to \pm\infty$ by first considering the reciprocal function. Noting that $u(g) \sim \frac{1}{3}g^2$ as $g \to +\infty$ one obtains $g(u) \sim \sqrt{3u}$ as $u \to \infty$. This corresponds to taking first $h \to 0^+$ for some $\epsilon > 0$ fixed, and then $\epsilon \to 0^+$, yielding $m \sim h^{1/3}\sqrt{3\epsilon h^{-2/3}} = \sqrt{3\epsilon}$, in agreement with equation (2). Similarly, $u(g) \sim -\frac{1}{g}$ as $g \to 0^+$ gives $g(u) \sim \frac{1}{-u}$ in $u \to -\infty$. Indeed the limit $h \to 0^+$ for some $\epsilon < 0$ fixed gives $m \sim h^{1/3}\frac{1}{-\epsilon h^{2/3}} = \frac{1}{-\epsilon}h$, i.e a finite susceptibility in the high temperature phase, that diverges with the exponent $\gamma = 1$ as $\epsilon \to 0^-$.



On the figure we have also plotted the curves $m(\epsilon, h)$ for three small values of h, obtained by solving without approximation the complete self-consistent equation. These are the same data as in the right panel of the first figure, but with a rescaling of the horizontal and vertical axis according to the scaling analysis. One sees indeed that in the limit of small h the curves collapse one over the other, and agree with the analytic prediction for $g(u)$.

3 General consequences of the existence of a scaling function

9. In $u = 0$ the scaling function g must go to a finite, strictly positive constant $g(0)$. Indeed at the critical temperature, i.e. for $\epsilon = 0$, that implies $u = 0$, the magnetization is a non-trivial function of h. One has then $m(\epsilon = 0, h) \sim h^a g(0) \propto h^a$ as $h \to 0^+$. By definition of the critical exponent δ one obtains $a = 1/\delta$.

10. If one takes the limit $h \to 0^+$ with $\epsilon > 0$ fixed then $u \to +\infty$. With the assumption $g(u) \propto u^c$ for $u \to +\infty$, one obtains $m \propto h^a(\epsilon h^{-b})^c = h^{a-bc}\epsilon^c$. For this expression to have a non-trivial limit as $h \to 0$ one must have $a - bc = 0$. Then $m \propto \epsilon^c$, taking the limit $\epsilon \to 0^+$ one recognizes the definition of the exponent β. Solving for the exponents b and c one obtains $c = \beta$ and $b = \frac{a}{c} = \frac{1}{\beta\delta}$.

11. Assume that $g(u) \propto (-u)^{c'}$ for $u \to -\infty$ for some exponent c'. If one takes the limit $h \to 0^+$ with $\epsilon < 0$ fixed then $u \to -\infty$ and $m \propto h^{a-bc'}(-\epsilon)^{c'}$. In this high temperature situation the magnetization must be proportional to h in the limit $h \to 0^+$ for any finite $\epsilon < 0$, as there is no spontaneous magnetization and the susceptibility is finite. Hence $a - bc' = 1$, which gives $c' = \frac{a-1}{b} = \beta(1-\delta)$ according to the values of a and b found previously. The factor $(-\epsilon)^{c'}$ multiplying h is the magnetic susceptibility, taking then the limit $\epsilon \to 0^-$ after $h \to 0^+$ one recongnizes that $c' = -\gamma$, so finally $\gamma = \beta(\delta - 1)$.

12. From $m \sim h^a g(\epsilon\, h^{-b})$ we obtain

$$\begin{aligned}\frac{\partial m}{\partial h} &\sim a h^{a-1} g(\epsilon\, h^{-b}) + h^a g'(\epsilon\, h^{-b})(-b)\epsilon\, h^{-b-1} \\ &\sim h^{a-1}\left(a g(\epsilon\, h^{-b}) - b\, \epsilon\, h^{-b} g'(\epsilon\, h^{-b})\right) ,\end{aligned}$$

which is indeed of the scaling form given in the text with $a' = a-1$ and $\widehat{g}(u) = \frac{1}{\delta} g(u) - \frac{1}{\beta\delta} u\, g'(u)$.

13. Assume that $\widehat{g}(u) \propto u^{c''}$ for $u \to +\infty$, which corresponds to the situation $\epsilon > 0$ and $h \to 0^+$. Then $\chi \propto h^{a-1}u^{c''} = h^{a-1-bc''}\epsilon^{c''}$ must be finite when $h \to 0^+$ with $\epsilon > 0$ fixed, which gives $a - 1 - bc'' = 0$ and thus $c'' = \beta(1-\delta)$. Taking then $\epsilon \to 0^+$ one recognize the definition of the exponent γ', hence $\gamma' = -c'' = \beta(\delta - 1)$.

14. The mean-field exponents are $\beta = 1/2$, $\delta = 3$, $\gamma = \gamma' = 1$, which satisfy the derived relations between exponents. Moreover the values of a and b used in the mean-field case to obtain a non-trivial scaling function do match with the generic relations $a = 1/\delta$, $b = \frac{1}{\beta\delta}$, and one can also check that the asymptotic behaviors of the scaling functions g and $\widehat{g}$ of the mean-field model agree with those obtained in the generic case.

4 Another relationship between critical exponents

15. In a ferromagnetic system with translation invariance, we have for all i that $\langle \sigma_i \rangle = m_{\rm sp}$. By linearity of the average, $\langle M(V) \rangle = L^d m_{\rm sp} \propto L^d \epsilon^\beta$.

16. We first derive an exact expression for the variance of the total magnetization in V, valid even in absence of translational invariance:

$$\mathrm{Var}[M(V)] = \langle M(V)^2 \rangle - \langle M(V) \rangle^2 = \sum_{i,j \in V} [\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle] = \sum_{i,j \in V} G_{i,j} \ .$$

We evaluate now its order of magnitude:

$$\mathrm{Var}[M(V)] = \sum_{i \in V} \sum_{j \in V} G(r_{ij}) \propto L^d \int_0^L r^{d-1} dr \frac{g(r/\xi)}{r^{d-2+\eta}},$$

where r_{ij} is the distance between the two sites i and j, and where we rewrote the sum as an integral since $L \gg 1$. As we are only looking for an order of magnitude we approximate the domain by a sphere of radius L, neglect the constant prefactors, and transform the sum over one of the sites into a multiplication by L^d. Now we can use that if $L \lesssim \xi$, the argument of the scaling function is $0 \leq r/\xi \leq 1$ where $g \sim \mathcal{O}(1)$. Computing the integral yields $\mathrm{Var}[M(V)] \propto L^{d+2-\eta}$.

17. If $L \sim \xi$, one has $\langle M(V) \rangle \sim \xi^d \epsilon^\beta \sim |\epsilon|^{\beta - \nu d}$ and $\sqrt{\mathrm{Var}[M(V)]} \sim \xi^{(d+2-\eta)/2} \sim |\epsilon|^{-\frac{\nu}{2}(d+2-\eta)}$. This yields $\beta = \frac{\nu}{2}(\eta + d - 2)$ if the mean and standard deviation of $M(V)$ are of the same order.

18. The mean-field exponents $\beta = 1/2$, $\eta = 0$ and $\nu = 1/2$ satisfy this relation in dimension $d = 4$, which is the upper critical dimension of the model. For $d < 4$ the relationship $\beta = \frac{\nu}{2}(\eta + d - 2)$ is satisfied, with exponents that depend on d and differ from their mean-field value. For $d > 4$ the exponents take their mean-field value, the relation $\beta = \frac{\nu}{2}(\eta + d - 2)$ is no longer satisfied.