ICFP M1 - Dynamical Systems and Chaos - TD n°1 - Exercises
Phase space and linear stability analysis

Baptiste Coquinot, Stephan Fauve
baptiste.coquinot@ens.fr

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1 Phase space: a simple example. Consider the simple dynamics of a mass attached to a spring ($k > 0$ and $m > 0$),

$$m\ddot{x} + kx = 0, \tag{1}$$

even though this system exhibits a trivial solution in terms of harmonic functions, we will here see how we can build intuition for the behavior of the solutions without explicitly solving for $x(t)$. This will become specially valuable when dealing with more complex dynamics for each analytical solutions are unnatainable.

1. The **state** of a deterministic dynamical system is composed of the set of variables that uniquely determines the future states of the system. In other words, the phase space variables evolve according to first order differential equations. Identify $\dot{x} = v$ to rewrite Eq. 1 as a system of first order ODEs.
2. The equations of motion assign a vector $(\dot{x}, \dot{v})$ at each point (x, v) in the phase plane, representing a **vector field**. Sketch the vector field as well as some example solutions in the phase-plane (x, v).
3. Find a parametric expression for the oscillatory orbits of the system. How does this relate to the notion of energy?

2 Phase space: general linear dynamics on the plane. Consider now the general setting of a system of linear ODEs,

$$\begin{cases} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2, \end{cases} \tag{2}$$

with general solution $\vec{x}(t) = e^{\lambda t}\vec{v}$

1. Find the characteristic equation to solve for λ_i in terms of the trace τ and the determinant Δ of the Jacobian $A_{ij} = \partial \dot{x}_i / \partial x_j$.
2. Assuming $\lambda_1 \neq \lambda_2$ nonzero eigenvalues, write the general solution to Eq. 2.
3. Draw the trajectories of the phase space with,
   (a) $\lambda_2 < -\lambda_1 < 0$ (**saddle node**)
   (b) $\lambda_2 < \lambda_1 < 0$ and $\lambda_1 > \lambda_2 > 0$ (**stable/unstable node**)
   (c) $\tau^2 - 4\Delta < 0$ and $\tau = 0$ (**center**)
   (d) $\tau^2 - 4\Delta < 0$ and $\tau \neq 0$ (**stable and unstable spiral**)
4. Discuss the case when $\lambda_1 = \lambda_2$.
5. Sketch a phase diagram in the $\tau - \Delta$ plane with the different behaviors of the solutions to Eq. 2.

3 Fixed points and linearization: a model of species competition. Consider the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{3}$$

and suppose that (x^*, y^*) is a fixed point: $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. Let $\epsilon_x = x - x^*$ and $\epsilon_y = y - y^*$ denote small perturbations away from the fixed point. To know whether the fixed point is stable, we focus on the vicinity of the fixed point and follow the evolution of such perturbations.

1. Derive $\dot{\epsilon}_x$ and $\dot{\epsilon}_y$ up to second order to find $\dot{\vec{\epsilon}} = \mathbf{A}\vec{\epsilon}$. We thus obtain the linearized dynamics in the vicinity of the fixed point, which we can analyse with the methods of Problem 1.

Consider now that x and y represents the population of small and big fish, respectively. Assume that the population of small fish grows at a constant rate μ while that of the big fish decays with a rate ν if they don't feed on the small fish.

2. Argue that under these assumptions $f(x, y) = \mu x - axy$ and $g(x, y) = -\nu y + bxy$. What do the constants a and b represent?
3. Find the fixed points of this system and analyze their linear stability. Interpret the fixed points.
4. Draw exemplar solutions in the phase space and discuss their qualitative behavior.
5. If fisherman are catching both species of fish at the same rate, how are their relative steady-state populations affected?