ICFP M1 - Dynamical Systems and Chaos - TD n°1 - Solutions
Phase space and linear stability analysis

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1 Phase space: a simple example. Consider the simple dynamics of a mass attached to a spring ($k > 0$ and $m > 0$),

$$m\ddot{x} + kx = 0, \tag{1}$$

even though this system exhibits a trivial solution in terms of harmonic functions, we will here see how we can build intuition for the behavior of the solutions without explicitly solving for $x(t)$. This will become specially valuable when dealing with more complex dynamics for each analytical solutions are unattainable.

1. The **state** of a deterministic dynamical system is composed of the set of variables that uniquely determines the future states of the system. In other words, the phase space variables evolve according to first order differential equations. Identify $\dot{x} = v$ to rewrite Eq. 1 as a system of first order ODEs.
2. The equations of motion assign a vector $(\dot{x}, \dot{v})$ at each point (x, v) in the phase plane, representing a **vector field**. Sketch the vector field as well as some example solutions in the phase-plane (x, v).
3. Find a parametric expression for the oscillatory orbits of the system. How does this relate to the notion of energy?

Correction

1. Identifying the velocity $v = \dot{x}$ we get,
$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x. \end{cases}$$
2. Just evaluate the vector $(\dot{x}, \dot{v})$ at a few examplar (x, v) points for intuition. The vector field winds around the origin in a clockwise fashion. At the origin we have a stable fixed point.
3. From 2), you've seen that the system admits oscillatory solutions that define a elliptical trajecotry in the phase space. To parameterize it, we need to find a relationship between x and v. In order to do that, we divide $\dot{v}$ by $\dot{x}$ and solve by separation of variables,
$$\begin{aligned} \frac{\dot{v}}{\dot{x}} = \frac{dv}{dx} &= \frac{-\frac{k}{m}x}{v} \\ mv\,dv &= -kx\,dx \\ \frac{mv^2}{2} &= -\frac{kx^2}{2} + E \\ E &= \frac{kx^2}{2} + \frac{mv^2}{2}, \end{aligned}$$
where we can see E corresponds to the energy of the system (kinetic + potential), which is a constant along the attractor of this Hamiltonian, energy-preserving, system.

2 Phase space: general linear dynamics on the plane. Consider now the general setting of a system of linear ODEs,

$$\begin{cases} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2, \end{cases} \tag{2}$$

with general solution $\vec{x}(t) = e^{\lambda t}\vec{v}$.

1. Find the characteristic equation to solve for λ_i in terms of the trace τ and the determinant Δ of the Jacobian $A_{ij} = \partial \dot{x}_i / \partial x_j$.
2. Assuming $\lambda_1 \neq \lambda_2$ nonzero eigenvalues, write the general solution to Eq. 2.
3. Draw the trajectories of the phase space with,

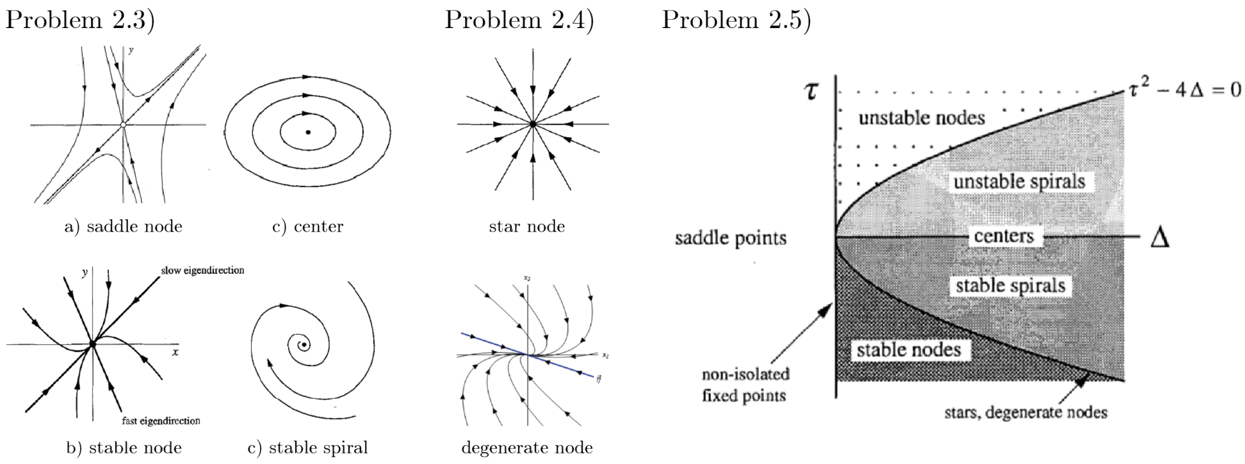

Figure 1: Solutions to problem 2). Figures extracted from [1].

(a) $\lambda_2 < -\lambda_1 < 0$ (**saddle node**)
(b) $\lambda_2 < \lambda_1 < 0$ and $\lambda_1 > \lambda_2 > 0$ (**stable/unstable node**)
(c) $\tau^2 - 4\Delta < 0$ and $\tau = 0$ (**center**)
(d) $\tau^2 - 4\Delta < 0$ and $\tau \neq 0$ (**stable and unstable spiral**)

4. Discuss the case when $\lambda_1 = \lambda_2$.

5. Sketch a phase diagram in the $\tau - \Delta$ plane with the different behaviors of the solutions to Eq. 2.

Correction

1. Substituting $\vec{x}(t) = e^{\lambda t}\vec{v}$ into $\dot{\vec{x}} = A\vec{x}$ we get,

$$\lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$$
$$\lambda\vec{v} = A\vec{v},$$

and so λ and $\vec{v}$ are the eigenvalues and eigenvectors of the Jacobian A and so $\vec{x}(t) = e^{\lambda t}\vec{v}$ is an eigensolution. Therefore we can write down the characteristic equation,

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0,$$

which solves to

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2},$$

where $\tau = a + d$ and $\Delta = ad - bc$ are the trace and the determinant respectively.

2. When $\lambda_1 \neq \lambda_2$ the corresponding eigenvectors are linearly independent, meaning that any initial condition can be written as a linear combination of eigenvectors $\vec{x}_0 = c_1\vec{v}_1 + c_1\vec{v}_2$, which allows us to write down the general solution as

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

$x(t)$ is the general solution because it is a linear combination of solutions to $\dot{\vec{x}} = A\vec{x}$ and thus is itself a solution. In addition, it satisfies the initial condition $\vec{x}(0) = \vec{x}_0$ and so by the existence and uniqueness theorem it is the *only* solution.

3. The solution is in Fig. 1. Note that trajectories become parallel to the slow eigendirection.

4. If the eigenvalues are the same, then there are either two linearly independent eigenvectors, or only one. The number of linearly independent eigenvectors associated with an eigenvalue λ for an $n \times n$ matrix is given by $\dim(Es) = n - \text{rank}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\mathbb{1}_n\right)$. If there are two linearly independent eigenvectors, they span the entire space and every vector is an eigenvector with the same eigenvalue $\lambda = \lambda_1 = \lambda_2$. In this case A must be a multiple of the identity matrix, $A = \lambda\mathbb{1}$. Then if $\lambda \neq 0$ all trajectories are straight lines through the origin and the fixed point is a **star or proper node**, Fig. 1. If $\lambda = 0$ then the whole phase space is filled with fixed points ($\dot{\vec{x}} = 0$). If there is only one eigenvector, then we call the node **degenerate or improper node**. In this case, additional linearly independent eigenvectors must be constructed to write down a general solution [1]. An example of a phase space portrait near an improper/degenerate node

[1] The general solution for a two-dimensional problem may be written as $\vec{x}(t) = c_1\vec{\eta}e^{\lambda t} + c_2\left(\vec{\eta}te^{\lambda t} + \vec{\rho}e^{\lambda t}\right)$; with $\vec{\rho}$ solving $\left[\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\mathbb{1}_n\right)\right]\vec{\rho} = \vec{\eta}$. You can check that two solutions are linearly independent by applying $\left[\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\mathbb{1}_n\right)\right]$ to $\vec{x}(t) = 0$. The structure of the second independent solution causes the quasi-spirals in Fig. 1: one follows the eigenvector $\vec{\eta}$ only in the long-time limit where the term $c_2t\vec{\eta}e^{\lambda t}$ dominates; at early times one follows different directions depending on the initial conditions.

is shown in Fig. 1. The imaginary part of λ vanishes for $\tau^2 - 4\Delta \geq 0$, hence the spiralling velocity of a (stable or unstable) focus vanishes continuously as $\tau^2 - 4\Delta \to 0^-$. I'd recommend plotting some solutions to get intuition.

5. See Fig. 1.

3 Fixed points and linearization: a model of species competition.

Consider the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{3}$$

and suppose that (x^*, y^*) is a fixed point: $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. Let $\epsilon_x = x - x^*$ and $\epsilon_y = y - y^*$ denote small perturbations away from the fixed point. To know whether the fixed point is stable, we focus on the vicinity of the fixed point and follow the evolution of such perturbations.

1. Derive $\dot{\epsilon}_x$ and $\dot{\epsilon}_y$ up to second order to find $\dot{\vec{\epsilon}} = \mathbf{A}\vec{\epsilon}$. We thus obtain the linearized dynamics in the vicinity of the fixed point, which we can analyse with the methods of Problem 1.

Consider now that x and y represents the population of small and big fish, respectively. Assume that the population of small fish grows at a constant rate μ while that of the big fish decays with a rate ν if they don't feed on the small fish.

2. Argue that under these assumptions $f(x, y) = \mu x - axy$ and $g(x, y) = -\nu y + bxy$. What do the constants a and b represent?
3. Find the fixed points of this system and analyze their linear stability. Interpret the fixed points.
4. Draw exemplar solutions in the phase space and discuss their qualitative behavior.
5. If fisherman are catching both species of fish at the same rate, how are their relative steady-state populations affected?

Correction

1. Let's derive for ϵ_x,

$$\begin{aligned}
\dot{\epsilon_x} &= \dot{x} && \text{(since } x^* \text{ is a constant)} \\
&= f(x^* + \epsilon_x, y^* + \epsilon_y) && \text{(by substitution)} \\
&= f(x^*, y^*) + \epsilon_x \frac{\partial f}{\partial x} + \epsilon_y \frac{\partial f}{\partial y} + \mathcal{O}(\epsilon_x^2, \epsilon_y^2, \epsilon_x \epsilon_y) && \text{(Taylor expand)} \\
&= \epsilon_x \frac{\partial f}{\partial x} + \epsilon_y \frac{\partial f}{\partial y} + \mathcal{O}(\epsilon_x^2, \epsilon_y^2, \epsilon_x \epsilon_y) && \text{(since} f(x^*, y^*) = 0)
\end{aligned}$$

The solution for ϵ_y is equivalent, yielding

$$\begin{pmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} + \text{(quadratic terms)}$$

2. $f(x, y) = x(\mu - ay)$ and $f(x, y) = y(bx - \nu)$. The growth the population of small fish is lower the larger the population of large fish, reducing the growth rate by an amount proportional to y, giving axy. Similarly, the population of large fish grows proportionally to the population of small fish x, resulting in the nonlinear term bxy. a thus represents how much the population of small fish decays when its predated and b represents how the population of large fish grows when it can successfully predate.

3. To find the fixed points, we compute

$$\begin{cases} \dot{x} = x(\mu - ay) = 0 \\ \dot{y} = y(-\nu + bx) = 0, \end{cases}$$

and so $(x_1^*, y_1^*) = (0, 0) \bigvee (x_2^*, y_2^*) = (\nu/b, \mu/a)$ are the fixed points. The linear stability is obtained by looking at the eigenvalues of the Jacobian matrix at the fixed points,

$$A = \begin{pmatrix} \mu - ay & -ax \\ by & bx - \nu \end{pmatrix}.$$

For $(x_1^*, y_1^*) = (0, 0)$ we have $A = \begin{pmatrix} \mu & 0 \\ 0 & -\nu \end{pmatrix}$, so the eigenvalues are $\lambda_1 = \mu$ and $\lambda_2 = -\nu$ and we are in presence of a saddle node: the flow points towards the original in the vertical direction and away from it in the horizontal direction. For $(x_2^*, y_2^*) = (\nu/b, \mu/a)$ we have $A = \begin{pmatrix} 0 & -a\nu/n \\ b\mu/a & 0 \end{pmatrix}$, which gives $\lambda_{1,2} = \pm i\sqrt{\mu\nu}$ which corresponds to a center, an oscillatory solution around (x_2^*, y_2^*).

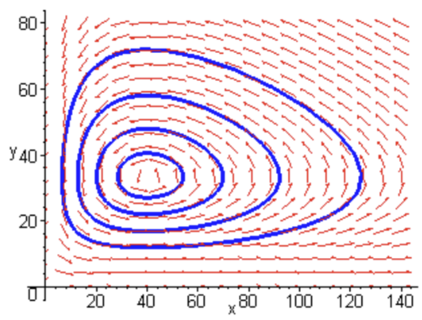

Figure 2: Vector field and example trajectories for the population dynamics of problem 3.4). In this case, $\mu > \nu$.

4. The relative steady-state populations are given by $\frac{x_2^*}{y_2^*} = \frac{a}{b}\frac{\nu}{\mu}$. If fisherman are catching both species at the same rate, that modifies the equations of motion to

$$\begin{cases} \dot{x} = x(\mu - \delta) - axy \\ \dot{y} = y(-\nu - \delta) + bxy, \end{cases}$$

implying that the steady-state populations are modified as,

$$\frac{x_2^*}{y_2^*} = \frac{a}{b}\frac{\nu}{\mu} \to \frac{a}{b}\frac{\nu + \delta}{\mu - \delta},$$

which increases with increasing δ. Therefore, if the fishing occurs at a constant rate δ for both species, the population of big fish y is impacted the most.

References

[1] Steven Strogatz, author. *Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering*. Second edition. Boulder, CO : Westview Press, a member of the Perseus Books Group, [2015], 2015.