ICFP M1 - Dynamical Systems and Chaos - TD n°2&3 - Exercises
The Method of Multiple Scales

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1 Weakly damped harmonic oscillator. We will start with an example that can be solved exactly to build intuition about the **method of multiple scales**. Consider the weakly damped linear oscillator ($0 < \epsilon \ll 1$),

$$\ddot{x} + 2\epsilon\dot{x} + \omega_0^2 x = 0, \tag{1}$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. Consider $\omega_0^2 = 1$ without loss of generality.

1. Show that the exact solution to this problem is $x(t,\epsilon) = (1-\epsilon^2)^{-1/2}e^{-\epsilon t}\sin\left[\sqrt{1-\epsilon^2}t\right]$. Identify the **two** main time scales of the system.
2. We will now introduce the method of multiple scales. Let us define a new slow timescale $T_1 = \epsilon t$, which we will assume to be constant with regard to the fast time scale $\tau = t$. In general, we can have N slow time scales and rewrite $x(t)$ as $x(\tau, T_1, \ldots, T_N)$. Show that up to $\mathcal{O}(\epsilon^2)$, we can rewrite Eq. 1 as,
$$\frac{\partial^2 x}{\partial \tau^2} + x = -2\epsilon\frac{dx}{d\tau} - 2\epsilon\frac{d^2 x}{d\tau dT_1} + \mathcal{O}(\epsilon^2)$$
3. Expand the solution to Eq. 1 as a series to find,
$$\frac{\partial^2 x_0}{\partial \tau^2} + \epsilon\left(\frac{\partial^2 x_1}{\partial \tau^2} + 2\frac{\partial^2 x_0}{\partial T_1 \tau}\right) + 2\epsilon\frac{\partial x_0}{\partial \tau} + x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2) = 0$$
4. Use the solvability condition at $\mathcal{O}(1)$ to find that $x_0 = A\sin\tau + B\cos\tau$. Note that the "constants" A and B are actually functions of the slow time scale T_1.
5. Determine $A(T_1)$ and $B(T_1)$ by going to next order in ϵ and getting rid of resonant terms.
6. Use the initial conditions to find the approximation,
$$x = e^{-\epsilon t}\sin t + \mathcal{O}(\epsilon).$$
7. Compare this with the exact solution of 1.1). The damping term actually has two effects: it dampens the oscillation amplitude, but also creates a shift in frequency. On what timescale do they occur?

2 Nonlinear oscillations: the van der Pol oscillator Consider the van der Pol oscillator,

$$\ddot{x} + (x^2 - \alpha)\dot{x} + \omega_0^2 x = 0 \tag{2}$$

1. Perform linear stability analysis in the vicinity of the stationary solutions and draw the phase space for $\alpha > 0$ and $\alpha < 0$.
2. Find the evolution equation for the energy of the oscillator $E = \frac{v^2}{2} + \frac{\omega_0^2 x^2}{2}$, where $v = \dot{x}$. Show that the origin is globally stable for $\alpha < 0$?
3. For $\alpha > 0$ linear stability analysis predicts exponential growth of the oscillations, but this is eventually stopped by the non-linear term $(x^2 - \alpha)$. Consider that for small $\alpha > 0$ (just after the bifurcation point) there is a nonlinear limit cycle that is very close to a linear harmonic oscillator, such that we can approximate $x(t)$ as $x(t) \sim a\sin(\omega t + \phi)$. Considering the energy balance over one cycle, determine how the amplitude of the oscillations a scales with α.
4. Show that for a periodic solution of Eq. 2 one has
$$\langle \dot{x}^2 \rangle = \omega_0^2 \langle x^2 \rangle, \tag{3}$$
where $\langle \cdot \rangle$ stands for time averaging.

5. A second role of the non-linearity is the production of higher harmonics and the deviation from the fundamental frequency. To see that, consider now a more general periodic solution of the form,

$$x(t) = \sum_n a_n \sin(n\omega t + \phi_n).$$

Give the expression of ω as a function of ω_0 and the amplitudes a_n of the harmonics. In what parameter range is the sinusoidal approximation used in question 2.3) valid?

6. We now consider the van der Pol oscillator but in the limit of large α. Make the change of variables $x(t) = \sqrt{\alpha}\frac{dy}{dt}$ in Eq. 2 and integrate the equation. Explain why the constant of integration can be taken equal to 0.

7. Defining $T = t/\alpha$, $u(T) = y(t)/\alpha$, $v = \frac{du}{dT}$, write down the differential equations for u and v.

8. Show that for $\alpha \to \infty$, the system involves two time scales. Find a relation $u = f(v)$ for which the two time scales are comparable.

9. Plot the **nullcline** $u = f(v)$ in the phase plane (u, v) and draw some example trajectories.

10. Compute the leading order approximation of the period of these relaxation oscillations in the limit $\alpha \to \infty$.