ICFP M1 - Dynamical Systems and Chaos - TD n°2&3 - Solutions
The Method of Multiple Scales

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1 Weakly damped harmonic oscillator. We will start with an example that can be solved exactly to build intuition about the **method of multiple scales**. Consider the weakly damped linear oscillator ($0 < \epsilon \ll 1$),

$$\ddot{x} + 2\epsilon\dot{x} + \omega_0^2 x = 0, \tag{1}$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. Consider $\omega_0^2 = 1$ without loss of generality.

1. Show that the exact solution to this problem is $x(t,\epsilon) = (1-\epsilon^2)^{-1/2} e^{-\epsilon t} \sin\left[\sqrt{1-\epsilon^2}t\right]$. Identify the **two** main time scales of the system.

   We can rewrite Eq. 1 as a system of first order ODEs in the $(x, \omega = \dot{x})$ phase space,

   $$\begin{cases} \dot{x} = \omega \\ \dot{\omega} = -x - 2\epsilon\omega, \end{cases}$$

   which we can solve with the methods of TD1. Identifying $A = \begin{pmatrix} 0 & 1 \\ -1 & -2\epsilon \end{pmatrix}$, we find that the eigenvalues $\lambda = -\epsilon \pm i\sqrt{1-\epsilon^2}$, from which we can already deduce that for $0 < \epsilon \ll 1$ the dynamics is a very slowly decaying spiral. To write down a full solution we need also the eigenvectors, which we can obtain by

   $$\begin{pmatrix} -\lambda & 1 \\ -1 & -2\epsilon - \lambda \end{pmatrix} \begin{pmatrix} v_x \\ v_\omega \end{pmatrix} = 0 \Rightarrow v_\omega = \lambda v_x.$$

   Recalling the results of TD1, we find,

   $$\begin{aligned} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix} &= c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2} \\ &= c_1 e^{(-\epsilon + i\sqrt{1-\epsilon^2})t} \begin{pmatrix} 1 \\ -\epsilon + i\sqrt{1-\epsilon^2} \end{pmatrix} + c_2 e^{(-\epsilon - i\sqrt{1-\epsilon^2})t} \begin{pmatrix} 1 \\ -\epsilon - i\sqrt{1-\epsilon^2} \end{pmatrix}. \end{aligned}$$

   Plugging in the initial conditions $x(0) = 0$ and $\omega(0) = 1$, we find,

   $$\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= (-\epsilon + i\sqrt{1-\epsilon^2})c_1 + (-\epsilon - i\sqrt{1-\epsilon^2})c_2 \end{aligned}$$

   which yields

   $$\begin{aligned} c_2 &= -c_1 \\ 1 &= \epsilon c_2 - i\sqrt{1-\epsilon^2}c_2 - \epsilon c_2 - i\sqrt{1-\epsilon^2}c_2 \Rightarrow c_2 = \frac{-1}{2i\sqrt{1-\epsilon^2}}. \end{aligned}$$

   Putting everything together we find,

   $$\begin{aligned} x(t) &= e^{-\epsilon t}\frac{1}{2i\sqrt{1-\epsilon^2}}\left[e^{i\sqrt{1-\epsilon^2}t} - e^{-i\sqrt{1-\epsilon^2}t}\right] \\ x(t) &= e^{-\epsilon t}\left(1-\epsilon^2\right)^{-1/2}\sin\left(\sqrt{1-\epsilon^2}t\right), \end{aligned}$$

   using Euler's formula. As expected, $x(t)$ exhibits a fast oscillatory time scale $t \sim \mathcal{O}(1)$ and a slower time scale $t \sim 1/\epsilon$ over which the amplitude of the oscillations decay.

2. We will now introduce the method of multiple scales. Let us define a new slow timescale $T_1 = \epsilon t$, which we will assume to be constant with regard to the fast time scale $\tau = t$. In general, we can have N slow time scales and rewrite $x(t)$ as $x(\tau, T_1, \ldots, T_N)$. Show that up to $\mathcal{O}(\epsilon^2)$, we can rewrite Eq. 1 as,

   $$\frac{\partial^2 x}{\partial \tau^2} + x = -2\epsilon\frac{dx}{d\tau} - 2\epsilon\frac{d^2x}{d\tau dT_1} + \mathcal{O}(\epsilon^2)$$

The time derivatives can be expanded using the chain rule, yielding

$$\dot{x} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T_1} + \mathcal{O}(\epsilon^2),$$

and,

$$\ddot{x} = \frac{\partial^2 x}{\partial \tau^2} + 2\epsilon \frac{\partial^2 x}{\partial \tau \partial T_1} + \mathcal{O}(\epsilon^2).$$

Plugging this into Eq. 1 we find,

$$\frac{\partial^2 x}{\partial \tau^2} + x = -2\epsilon \frac{\partial^2 x}{\partial \tau \partial T_1} - 2\epsilon \frac{\partial x}{\partial \tau} + \mathcal{O}(\epsilon^2)$$

3. Expand the solution to Eq. 1 as a series to find,

$$\frac{\partial^2 x_0}{\partial \tau^2} + \epsilon \left(\frac{\partial^2 x_1}{\partial \tau^2} + 2 \frac{\partial^2 x_0}{\partial T_1 \tau} \right) + 2\epsilon \frac{\partial x_0}{\partial \tau} + x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2) = 0$$

Writing $x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon_2 x_2(t) + \ldots$, we get,

$$\frac{\partial^2 x_0}{\partial \tau^2} + x_0 + \epsilon \frac{\partial^2 x_1}{\partial \tau^2} + \epsilon x_1 = -2\epsilon \frac{\partial^2 x_0}{\partial \tau \partial T_1} - 2\epsilon \frac{\partial x_0}{\partial \tau} + \mathcal{O}(\epsilon^2)$$
$$\frac{\partial^2 x_0}{\partial \tau^2} + \epsilon \left(\frac{\partial^2 x_1}{\partial \tau^2} + 2\epsilon \frac{\partial^2 x_0}{\partial \tau d T_1} \right) + 2\epsilon \frac{\partial x_0}{\partial \tau} + x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2) = 0.$$

4. Use the solvability condition at $\mathcal{O}(1)$ to find that $x_0 = A \sin \tau + B \cos \tau$. Note that the "constants" A and B are actually functions of the slow time scale T_1.

Collecting powers of ϵ yields a pair of differential equations

$$\mathcal{O}(1) : \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0$$
$$\mathcal{O}(\epsilon) : \frac{\partial^2 x_1}{\partial \tau^2} + 2 \frac{\partial^2 x_0}{\partial \tau \partial T_1} + 2 \frac{\partial x_0}{\partial \tau} + x_1 = 0,$$

From which we can see that at $\mathcal{O}(\epsilon)$ we have a simple harmonic oscillator equation with solution $x_0 = A \sin \tau + B \cos \tau$.

5. Determine $A(T_1)$ and $B(T_1)$ by going to next order in ϵ and getting rid of resonant terms.

Replacing x_0 into the $\mathcal{O}(\epsilon)$ equation we have the terms,

$$\frac{\partial x_0}{\partial \tau} = A \cos \tau - B \sin \tau$$
$$\frac{\partial^2 x_0}{\partial T_1 \partial \tau} = \partial_{T_1} A \cos \tau - \partial_{T_1} B \sin \tau,$$

yielding,

$$\frac{\partial^2 x_1}{\partial \tau^2} + 2 \left(\partial_{T_1} A \cos \tau - \partial_{T_1} B \sin \tau \right) + 2 \left(A \cos \tau - B \sin \tau \right) + x_1 = 0$$
$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2(\partial_{T_1} A + A) \cos \tau + 2(\partial_{T_1} B + B) \sin \tau,$$

We get rid of resonant terms to avoid the blowing up of solutions [1]. In order to do that, we need to set $\partial_{T_1} A + A = 0$ and $\partial_{T_1} B + B = 0$, yielding,

$$A(T_1) = A(0) e^{-T_1}$$
$$A(T_2) = A(0) e^{-T_2}.$$

6. Use the initial conditions to find the approximation,

$$x = e^{-\epsilon t} \sin t + \mathcal{O}(\epsilon).$$

[1] Aside on eliminating the resonant terms. The general solution to the resonantly forced oscillator $\ddot{x} + \omega_0^2 x = f e^{i\omega t} + \text{c.c}$, $f \in \mathbb{C}$ independent of t, is given by $x(t) = \left(C - \frac{ift}{2\omega} \right) e^{i\omega t} + \text{c.c.}$. A term with a polynomial (here linear) growth with time is called *secular*. After a time $t = \mathcal{O}(1/\epsilon)$, the expansion breaks down and our approximation ceases to be valid.

The initial condition for x yields $x(0) = 0 = x_0(0,0) + \epsilon x_1(0,0) + \mathcal{O}(\epsilon^2)$, which requires $x_0(0,0) = 0$ and $x_1(0,0) = 0$. Similarly, for $\dot{x}$ we have,

$$\dot{x}(0) = 1 = \frac{\partial x_0(0,0)}{\partial \tau} + \epsilon\left(\frac{\partial x_0(0,0)}{\partial T_1} + \frac{\partial x_1(0,0)}{\partial \tau}\right) + \mathcal{O}(\epsilon^2),$$

so

$$\partial_\tau x_0(0,0) = 1$$

and

$$\partial_{T_1} x_0(0,0) + \partial_\tau x_1(0,0) = 0.$$

Using the solution $x_0 = A\sin\tau + B\cos\tau$, we find from $x_0(0,0) = 0$ that $B(0) = 0$ which implies $B(T) = 0$, and from $\partial_\tau x_0(0,0) = 1$ that $A(0) = 1$ and so $A(T_1) = e^{-T_1}$ which yields,

$$x_0(\tau, T_1) = e^{-T_1}\sin T_1,$$

and therefore,

$$x(t) = e^{-\epsilon t}\sin t + \mathcal{O}(\epsilon)$$

7. Compare this with the exact solution of 1.1). The damping term actually has two effects: it dampens the oscillation amplitude, but also creates a shift in frequency. On what timescale do they occur?

   Comparing the two solutions we see that the approximate solution fails to capture the $(1-\epsilon^2)^{-1/2}$ correction and, more importantly, the manner in which the frequency of the oscillations is shifted slightly from 1. The frequency of the oscillations is $\sqrt{1-\epsilon^2} \approx 1 - \frac{\epsilon^2}{2}$, which means that after a very long time $t \sim \mathcal{O}(1/\epsilon^2)$ this frequency error will have a large cumulative effects. This is, in fact, a third super-slow time scale, which we could obtain either through higher order corrections in ϵ, or by introducing a slower time scale $T_2 = \epsilon^2 t$ to investigate the long-term phase shift caused by the $\mathcal{O}(\epsilon^2)$ error in frequency.

2 Nonlinear oscillations: the van der Pol oscillator

Consider the van der Pol oscillator,

$$\ddot{x} + (x^2 - \alpha)\dot{x} + \omega_0^2 x = 0 \tag{2}$$

1. Perform linear stability analysis in the vicinity of the stationary solutions and draw the phase space for $\alpha > 0$ and $\alpha < 0$.

   Rewriting Eq. 2 as a system of first order ODEs, we get,

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\omega_0^2 x - (x^2-\alpha)v \end{cases},$$

   which admit a fixed point at the origin $(x^*, v^*) = (0,0)$. Linearizing about this fixed point we find the jacobian matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & \alpha, \end{pmatrix}$$

   with eigenvalues $\lambda = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \omega_0^2}$. Thus, for $\frac{\alpha^4}{4} > \omega_0^2$ we have real eigenvalues and the fixed point is a stable or unstable node depending on the sign of λ. For $\frac{\alpha^4}{4} < \omega_0^2$ the origin is the focus of a spiral, which is stable from $\alpha < 0$ and unstable for $\alpha > 0$ (Hopf bifurcation).

2. Find the evolution equation for the energy of the oscillator $E = \frac{v^2}{2} + \frac{\omega_0^2 x^2}{2}$, where $v = \dot{x}$. Show that the origin is globally stable for $\alpha < 0$?

   We can write $\dot{E} = \dot{v}v + \omega_0^2 x\dot{x} = (\alpha - x^2)\dot{x}^2$. The origin is globally stable when $\alpha < 0$ because $\dot{E} < 0$: the system dissipates energy over time and eventually decays towards the origin.

3. For $\alpha > 0$ linear stability analysis predicts exponential growth of the oscillations, but this is eventually stopped by the non-linear term $(x^2 - \alpha)$. Consider that for small $\alpha > 0$ (just after the bifurcation point) there is a nonlinear limit cycle that is very close to a linear harmonic oscillator, such that we can approximate $x(t)$ as $x(t) \sim a\sin(\omega t + \phi)$. Considering the energy balance over one cycle, determine how the amplitude of the oscillations a scales with α.

   As time goes one, the energy is either growing or decaying depending on the sign of $(\alpha - x^2)$. For $x < \sqrt{\alpha}$, the energy is positive and the oscillations grow, and when $x > \sqrt{\alpha}$ the energy is negative and the balance between excitation for low x and dissipation for large x leads to a periodic solution. Nonetheless, the energy balance over one cycle requires that over a period T

$$\frac{1}{T}\int_0^T \dot{E}dt = 0$$
$$\alpha\langle\dot{x}^2\rangle - \langle x^2\dot{x}^2\rangle = 0$$
$$\alpha\omega^2 a^2\langle\cos^2(\omega t+\phi)\rangle - \omega^2 a^4\langle\sin^2(\omega t+\phi)\cos^2(\omega t+\phi)\rangle = 0,$$

using the fact that averaged over one cycle $\langle \cos^2(\omega t+\phi)\rangle = 1/2$ and $\langle \sin^2(\omega t+\phi)\cos^2(\omega t+\phi)\rangle = 1/8$, we find,

$$\begin{aligned} \alpha\frac{1}{2} - a^2\frac{1}{8} &= 0 \\ a &= 2\sqrt{\alpha}. \end{aligned}$$

Another way to find the relationship between a and α is to plug in $x(t) = a\sin(\omega t+\phi)$ into Eq. 2. We get $\dot{x} = \omega a\cos(\omega t+\phi)$ and $\ddot{x} = -\omega^2 a\sin(\omega t+\phi)$. Using the trigonometric identities $\sin^2(\omega t+\phi) = \frac{1}{2}\left[1-\cos(2\omega t+2\phi)\right]$ and $\cos(2\omega t+2\phi)\cos(\omega t+\phi) = \frac{1}{2}\left[\cos(3\omega t+3\phi)+\cos(\omega t+\phi)\right]$ we rewrite Eq. 2 as,

$$\begin{aligned} -\omega^2 a\sin(\omega t+\phi) + \left(a^2\sin^2(\omega t+\phi)-\alpha\right)\omega a\cos(\omega t+\phi) + \omega_0^2 a\sin(\omega t+\phi) &= 0 \\ -\omega^2 a\sin(\omega t+\phi) + \frac{a^3}{2}\omega\cos(\omega t+\phi) - \frac{a^2}{4}\left[\cos(3\omega t+3\phi)+\cos(\omega t+\phi)\right] & \\ -\alpha\omega a\cos(\omega t+\phi) + \omega_0^2 a\sin(\omega t+\phi) &= 0 \\ \sin(\omega t+\phi)(\omega_0^2-\omega^2) + \omega\cos(\omega t+\phi)\left[\frac{a^2}{4}-\alpha\right] - \frac{\omega^2 a^2}{4}\cos(3\omega t+3\phi) &= 0. \end{aligned}$$

Truncating to eliminate the mode-three term we find $\omega \approx \omega_0$ and $a \approx 2\sqrt{\alpha}$. So the amplitude of the oscillations scales as $a \sim \sqrt{\alpha}$, which we couldn't predict simply from linear stability analysis of the fixed point which predicted only exponential divergence of trajectories.

4. Show that for a periodic solution of Eq. 2 one has

$$\langle \dot{x}^2\rangle = \omega_0^2\langle x^2\rangle, \tag{3}$$

where $\langle\cdot\rangle$ stands for time averaging.

Multiplying Eq. 2 by x and taking the average over one period we find,

$$\begin{aligned} x\ddot{x} + x^3\dot{x} - \alpha\dot{x}x + \omega_0^2 x^2 &= 0 \\ \langle x\ddot{x}\rangle &= -\omega_0^2\langle x^2\rangle, \end{aligned}$$

since the averages over one period of terms involving $\dot{x}$ vanish. Integrating the l.h.s by parts we get,

$$\begin{aligned} x\frac{1}{T}\int_0^T \ddot{x}dt - \frac{1}{T}\int_0^T\left(\frac{dx}{dt}\right)^2 dt &= -\omega_0^2\langle x^2\rangle \\ \langle \dot{x}^2\rangle &= \omega_0^2\langle x^2\rangle. \end{aligned}$$

Notably, this result is formally identical to the virial theorem (avg. kinetic energy = avg. potential energy) found for simple harmonic oscillations, but we have shown that it also applies to oscillations in the presence of a velocity and amplitude-dependent force as described by the van der Pol oscillator.

5. A second role of the non-linearity is the production of higher harmonics and the deviation from the fundamental frequency. To see that, consider now a more general periodic solution of the form,

$$x(t) = \sum_n a_n\sin(n\omega t+\phi_n).$$

Give the expression of ω as a function of ω_0 and the amplitudes a_n of the harmonics. In what parameter range is the sinusoidal approximation used in question 2.3) valid?

We substitute the periodic ansatz $x(t) = \sum_n a_n\sin(n\omega t+\phi_n)$ into the virial equation to find,

$$\sum_{n,m} a_n a_m n\omega m\omega\langle\cos(n\omega t+\phi_n)\cos(m\omega t+\phi_m)\rangle = \omega_0^2\sum_{n,m} a_n a_m\langle\sin(n\omega t+\phi_n)\sin(m\omega t+\phi_m)\rangle.$$

The averages of the trignometric functions are given by,

$$\begin{aligned} \langle\cos(n\omega t+\phi_n)\cos(m\omega t+\phi_m)\rangle &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos(n\omega t+\phi_n)\cos(m\omega t+\phi_m)dt \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{1}{2}\left(\cos\left[(n+m)\omega t+\phi_n+\phi_m\right]+\cos\left[(n-m)\omega t+\phi_n-\phi_m\right]\right)dt \\ &= \frac{1}{2}\delta_{n,m}, \end{aligned}$$

and similarly,

$$\begin{aligned} \langle\sin(n\omega t+\phi_n)\sin(m\omega t+\phi_m)\rangle &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sin(n\omega t+\phi_n)\sin(m\omega t+\phi_m)dt \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{1}{2}\left(-\cos\left[(n+m)\omega t+\phi_n+\phi_m\right]+\cos\left[(n-m)\omega t+\phi_n-\phi_m\right]\right)dt \\ &= \frac{1}{2}\delta_{n,m}. \end{aligned}$$

Thus we find,

$$\omega = \omega_0 \sqrt{\frac{\sum_n a_n^2}{\sum_n a_n^2 n^2}}.$$

Therefore, when there is only one mode we recover the solution of 2.3), but when higher harmonics are excited we have $\omega < \omega_0$. The approximation of 2.3) is thus valid whenever $\sum_{n=2}^{\infty} a_n^2 n^2 \ll a_1^2$, so when one mode clearly dominates.

6. We now consider the van der Pol oscillator but in the limit of large α. Make the change of variables $x(t) = \sqrt{\alpha}\frac{dy}{dt}$ in Eq. 2 and integrate the equation. Explain why the constant of integration can be taken equal to 0.

Substituting $x = \sqrt{\alpha}\dot{y}$ we obtain,

$$\frac{d}{dt}\left[\frac{d^2y}{dt^2} + \alpha\left(\frac{1}{3}\left(\frac{dy}{dt}\right)^3\right) - \frac{dy}{dt} + \omega_0^2 y\right] = 0,$$

which integrates to,

$$\frac{d^2y}{dt^2} + \alpha\left(\frac{1}{3}\left(\frac{dy}{dt}\right)^3 - \frac{dy}{dt}\right) + \omega_0^2 y = C \in \mathbb{R}.$$

As a second-order ODE, the original van der Pol equation needs to be supplemented with two initial conditions for a unique solution, $x(0) = \frac{dy}{dt}|_{t=0} = x_0$ and $v(0) = \frac{d^2y}{dt^2}|_{t=0} = v_0$. Since y can be shifted by any constant without changing x for any x_0 and v_0 we may redefine $y \to y - \frac{1}{\omega_0}\left[v_0 + \lambda(x_0^3/3 - x_0)\right]$ and therefore ser $C = 0$ without loss of generality.

7. Defining $T = t/\alpha$, $u(T) = y(t)/\alpha$, $v = \frac{du}{dT}$, write down the differential equations for u and v.

With a slow time scale $T = t/\alpha$, $u = y/\alpha$, and $v(T) = \frac{du}{dT}$ we get,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dT}{dt}\frac{dy}{dT} = \frac{1}{\alpha}\frac{d(u\alpha)}{dT} = v(T)\\ \frac{d^2y}{dt^2} &= \frac{1}{\alpha}\frac{dv}{dT},\end{aligned}$$

Plugging this into,

$$\frac{d^2y}{dt^2} + \alpha\left(\frac{1}{3}\left(\frac{dy}{dt}\right)^3 - \frac{dy}{dt}\right) + \omega_0^2 y = 0,$$

we get,

$$\frac{1}{\alpha^2}\frac{dv}{dT} + \left(\frac{1}{3}v^2 - 1\right)v + \omega_0^2 u = 0,$$

8. Show that for $\alpha \to \infty$, the system involves two time scales. Find a relation $u = f(v)$ for which the two time scales are comparable.

The ODEs for v and u are,

$$\begin{aligned}\frac{dv}{dT} &= -\alpha^2\left\{\left(\frac{1}{3}v^2 - 1\right)v + \omega_0^2 u\right\}\\ \frac{du}{dT} &= v.\end{aligned}$$

So the time derivative of v is of order $v\alpha^2$, whereas the time derivative of u is only of order v, so v changes infinitely faster than u except if $\left(\frac{1}{3}v^2 - 1\right)v - \omega_0^2 u = 0$, which is the case when $\{\cdot\} = 0$,

$$u = \frac{1}{\omega_0^2}\left(v - \frac{1}{3}v^3\right).$$

Another way of deriving the same result is to note that there are two timescales, and write $u = u(\tau = T\alpha^2, T)$, $v = v(\tau = T\alpha^2, T)$. The derivatives transform as,

$$\frac{d}{dT} = \frac{\partial}{\partial T} + \alpha^2\frac{\partial}{\partial \tau},$$

Substituting into the ODEs we find,

$$\begin{aligned}\frac{1}{\alpha^2}\partial_T v + \partial_\tau v + \left(\frac{1}{3}v^2 - 1\right)v + \omega_0^2 u = 0\\ \partial_T u + \alpha^2\partial_\tau u = v.\end{aligned}$$

The second equation gives, at leading order $\mathcal{O}(\alpha^2)$ that $\partial_\tau u = 0$, hence u doesn't depend on the fast time scale τ.

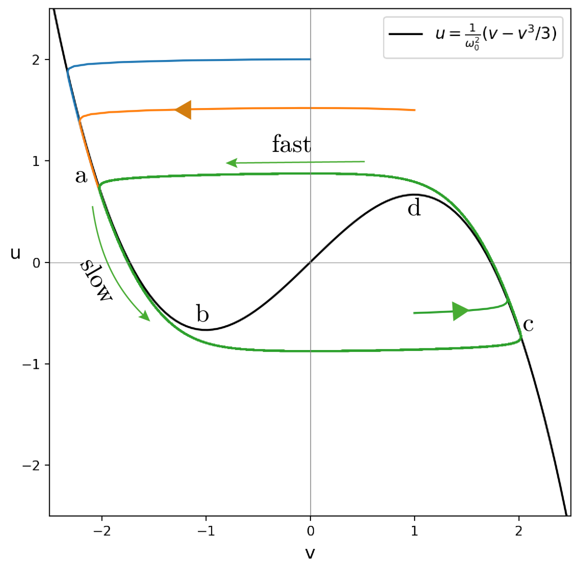

Figure 1: Example trajectories of the van der Pol oscillator in the (u, v) plane. The cubic nullcline determines much of the dynamics: starting from any initial condition, the trajectory moves fast in the v direction towards the nullcline (let's say to point a), then it slowly moves along the nullcline before zapping back from b to c. This motion is repeated periodically resulting in limit cycle dynamics.

9. Plot the **nullcline** $u = f(v)$ in the phase plane (u, v) and draw some example trajectories.

   See Fig. 1.

10. Compute the leading order approximation of the period of these relaxation oscillations in the limit $\alpha \to \infty$. In the limit $\alpha \to \infty$ the dominant contribution to the period is coming from the trajectories along the nullcline $u = u_{nc} = f(v) = \frac{1}{\omega_0^2}(v - v^3/3)$.

$$
\begin{aligned}
T &= 2\int_{T(a)}^{T(b)} dT \\
&= 2\int_{-\frac{2}{3\omega_0^2}}^{\frac{2}{3\omega_0^2}} \frac{dT}{du_{nc}} du_{nc} \\
&= 2\int_{2}^{1} \frac{1}{v}\frac{df}{dv} dv \\
T &= \frac{3 - 2\log(2)}{\omega_0^2}.
\end{aligned}
$$

To determine the limits of the integrals we first identity the values of u corresponding to the extrema of $f(v)$ and then find the corresponding v values. Also, we us the fact that $\frac{du}{dT} = v$ to rewrite $\frac{dT}{du_{nc}} du_{nc} = \frac{1}{v}\frac{df}{dv} dv$. This result is consistent dimensionally since we made the change of variables $T = t/\alpha$, hence in terms in terms of dimensions $[T] = \text{time}^2$ since $[\alpha] = \text{time}^{-1}$.