ICFP M1 - Dynamical Systems and Chaos - TD n°4 - Exercises
Nonlinear oscillators

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1 Instability of linearly coupled oscillators.

Consider two linearly coupled oscillators, with

$$\begin{cases} \ddot{x} = -\omega_1^2 x + ay \\ \ddot{y} = -\omega_2^2 y + bx, \end{cases}$$

with eigenfrequencies ω_i, $|\omega_1 - \omega_2| \ll \omega_1, \omega_2$, and a and b are coupling constants.

1. Look for solutions of the form $x(t) = e^{\lambda t}\hat{x}$, $y(t) = e^{\lambda t}\hat{y}$ and find an expression for λ_i^2 $(i = 1, 2)$
2. Consider $0 < ab \ll 1$. Classify the behavior of the solutions and show that the difference between the two oscillation frequencies is increased by the strength of the coupling.
3. Consider now $ab < 0$ and small. Show that the system is unstable for a critical value of $|ab|$ that depends on ω_i. What happens when $\omega_1 = \omega_2$?
4. Take $\omega_1 = \omega_0$, $\omega_2 = \omega_0 + \epsilon\delta$, $a = \epsilon\alpha$, $b = \epsilon\beta$ where $\epsilon \ll 1$ and $\omega_0, \delta, \alpha, \beta$ are of $\mathcal{O}(1)$. Define a slow time scale $T = \epsilon\tau$ (coming from ω_2), and rewrite x and y as $x(t) = u(t, T)$, $y(t) = v(t, T)$. Expanding u and v to first order in ϵ, show that the solvability condition yields,
$$\partial_T^2 A - i\delta\partial_T A + \frac{\alpha\beta}{4\omega_0^2} A = 0, \tag{1}$$
for the amplitude of the u oscillations.
5. Give the conditions in δ, α, β and ω_0 for instability. Compare with question 1.3).
6. We now consider the two coupled nonlinear oscillators,
$$\begin{cases} \ddot{x} = -\omega_1^2 \sin x + ay \\ \ddot{y} = -\omega_2^2 \sin y + bx. \end{cases}$$
Close to the onset of instability, the trajectories slowly spiral out of the origin we can derive write,
$$x(t) = u(t, T) \sim A(T)e^{i\omega_0 t} + A^*(T)e^{-i\omega_0 t} + \mathcal{O}(\epsilon),$$
where $A(T)$ is the slowly varying amplitude of the oscillations. Assuming that $\partial_T^2 A(T)$ can be written in terms of A, $\partial_T A$ and their complex conjugates, and using symmetry arguments, show that the amplitude equation is of the form,
$$\partial_T^2 A = \mu A + i\nu\partial_T A + \gamma A^2 A^*, \tag{2}$$
where μ, ν and γ are real constants.
7. Show that the second term in Eq. 2 can be removed with a simple change of variables.
8. Give the condition on γ for the existence of finite amplitude stationary solutions for $\mu > 0$.