ICFP M1 - Dynamical Systems and Chaos - TD n°4 - Solutions
Nonlinear oscillators

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1 Instability of linearly coupled oscillators. Consider two linearly coupled oscillators, with

$$\begin{cases} \ddot{x} = -\omega_1^2 x + ay \\ \ddot{y} = -\omega_2^2 y + bx, \end{cases}$$

with eigenfrequencies ω_i, $|\omega_1 - \omega_2| \ll \omega_1, \omega_2$, and a and b are coupling constants.

1. Look for solutions of the form $x(t) = e^{\lambda t}\hat{x}$, $y(t) = e^{\lambda t}\hat{y}$ and find an expression for λ_i^2 $(i = 1, 2)$

   Plugging in the ansatz we get,

$$\begin{cases} \lambda^2 \hat{x} = -\omega_1^2 \hat{x} + a\hat{y} \\ \lambda^2 \hat{y} = -\omega_2^2 \hat{y} + b\hat{x} \end{cases}$$

   with nontrivial solution $\lambda^4 + \lambda^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 - ba = 0$,

$$\lambda^2 = -\frac{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4ab}}{2}.$$

2. Consider $0 < ab \ll 1$. Classify the behavior of the solutions and show that the difference between the two oscillation frequencies is increased by the strength of the coupling.

   For $0 < ab \ll 1$ we find, $\lambda^2 \sim -\frac{\omega_1^2+\omega_2^2}{2}$ since $|\omega_1 - \omega_2| \ll \omega_1 + \omega_2$. Therefore the eigenvalues are purely imaginary and we have a center. In addition, increasing the strength of the coupling ab, results in a larger $|\lambda_1 - \lambda_2|$, so the oscillations frequencies are increased. (Note that actually the system can become unstable if we allow $\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4ab} > (\omega_1^2 + \omega_2^2)$.)

3. Consider now $ab < 0$ and small. Show that the system is unstable for a critical value of $|ab|$ that depends on ω_i. What happens when $\omega_1 = \omega_2$?

   For $ab < 0$, there is a critical point when $ab = -(\omega_1^2 - \omega_2^2)^2/4$, in which case we only have two degenerate eigenvalues: two pairs of oscillation frequencies coincide. For $|ab| > \frac{(\omega_1^2-\omega_2^2)^2}{4}$ we get $\lambda^2 = -\alpha^2 \pm i\beta^2$ s.t. two of the four roots have positive real parts indicating instability. When $\omega_1 = \omega_2$, then the system is unstable whenever $ab < 0$.

4. Take $\omega_1 = \omega_0$, $\omega_2 = \omega_0 + \epsilon\delta$, $a = \epsilon\alpha$, $b = \epsilon\beta$ where $\epsilon \ll 1$ and $\omega_0, \delta, \alpha, \beta$ are of $\mathcal{O}(1)$. Define a slow time scale $T = \epsilon\tau$ (coming from ω_2), and rewrite x and y as $x(t) = u(t, T)$, $y(t) = v(t, T)$. Expanding u and v to first order in ϵ, show that the solvability condition yields,

$$\partial_T^2 A - i\delta\partial_T A + \frac{\alpha\beta}{4\omega_0^2} A = 0, \tag{1}$$

   for the amplitude of the u oscillations.

   Redefining the variables we get, and transforming the derivatives accordingly, $\frac{d}{dt} = \partial_t + \epsilon\partial T$, we find

$$\begin{cases} \partial_t^2 x + 2\epsilon\partial_{tT} x = -\omega_0^2 u + \epsilon\alpha y \\ \partial_t^2 y + 2\epsilon\partial_{tT} y = -\omega_0^2 y - 2\epsilon\delta\omega_0 y + \epsilon\beta x. \end{cases}$$

   Writing $x(t) = u(t, T) \sim u_0(t, T) + \epsilon u_1(t, T)$ and $y(t) = v(t, T) \sim v_0(t, T) + \epsilon v_1(t, T)$ we get,

$$\begin{cases} \partial_t^2 u_0 + \epsilon\partial_t^2 u_1 + 2\epsilon\partial_{tT} u_0 = -\omega_0^2 u_0 - \epsilon\omega_0^2 u_1 + \epsilon\alpha v_0 + \mathcal{O}(\epsilon^2) \\ \partial_t^2 v_0 + \epsilon\partial_t^2 v_1 + 2\epsilon\partial_{tT} v_0 = -\omega_0^2 v_0 - \epsilon\omega_0^2 v_1 - 2\epsilon\delta\omega_0 v_0 + \epsilon\beta u_0. \end{cases}$$

   Collecting terms at different orders of ϵ we find that,

$$\mathcal{O}(1) : \begin{cases} \partial_t^2 u_0 + \omega_0^2 u_0 = 0 \\ \partial_t^2 v_0 + \omega_0^2 u_0 = 0 \end{cases}$$

$$\mathcal{O}(\epsilon) : \begin{cases} \partial_t^2 u_1 + \omega_0^2 u_1 = \alpha v_0 - 2\partial_{tT} u_0 \\ \partial_t^2 v_1 + \omega_0^2 v_1 = -2\partial_{tT} v_0 - 2\delta v_0 \omega_0 + \beta u_0 \end{cases} .$$

At first order we have uncoupled harmonic oscillator with frequency ω_0,

$$u_0 = A(T)e^{i\omega_0 t} + \text{c.c.}$$
$$v_0 = B(T)e^{i\omega_0 t} + \text{c.c.}.$$

Substituting the $\mathcal{O}(1)$ solution into the $\mathcal{O}(\epsilon)$ equations and eliminating resonant terms we find

$$-2i\omega_0 \partial_T A + \alpha B = 0$$
$$-2\omega_0 \delta B - 2i\omega_0 \partial_T B + \beta A = 0,$$

and thus

$$\partial_T^2 A - i\delta \partial_T A + \frac{\alpha\beta}{4\omega_0^2} A = 0.$$

5. Give the conditions in δ, α, β and ω_0 for instability. Compare with question 1.3).

The exponential ansatz $A \sim e^{\lambda t}$ leads to,

$$\lambda^2 - i\delta\lambda + \frac{\alpha\beta}{4\omega_0^2} = 0$$
$$\lambda = \frac{i\delta \pm \sqrt{-\delta^2 - \alpha\beta/\omega_0^2}}{2} = \frac{i\delta}{2} \pm \frac{i}{2}\sqrt{\delta + \frac{\alpha\beta}{\omega_0^2}}.$$

Thus, an instability occurs when $\alpha\beta < -\delta\omega_0^2$ so there is a critical point at $\alpha\beta = -\delta\omega_0^2$. This is in agreement with 1.3) since the critical point is at

$$\begin{aligned} ab &= -\frac{1}{4}\left(\omega_1^2 - \omega_2^2\right)^2 \\ &= -\frac{1}{4}\left(2\epsilon\delta\omega + \mathcal{O}(\epsilon^2)\right)^2 \\ &= -\epsilon^2\delta^2\omega_0^2 + \text{h.o.t} \\ &= \epsilon^2\alpha\beta \\ &= ab \end{aligned}$$

6. We now consider the two coupled nonlinear oscillators,

$$\begin{cases} \ddot{x} = -\omega_1^2 \sin x + ay \\ \ddot{y} = -\omega_2^2 \sin y + bx. \end{cases}$$

Close to the onset of instability, the trajectories slowly spiral out of the origin we can derive write,

$$x(t) = u(t,T) \sim A(T)e^{i\omega_0 t} + A^*(T)e^{-i\omega_0 t} + \mathcal{O}(\epsilon),$$

where $A(T)$ is the slowly varying amplitude of the oscillations. Assuming that $\partial_T^2 A(T)$ can be written in terms of A, $\partial_T A$ and their complex conjugates, and using symmetry arguments, show that the amplitude equation is of the form,

$$\partial_T^2 A = \mu A + i\nu\partial_T A + \gamma A^2 A^*, \tag{2}$$

where μ, ν and γ are real constants.

The equations of x and y exhibit two symmetries: a) time-translation invariance $t \to t + \theta$ (it's an autonomous system) and b) time-reversal symmetry $t \to -t$ (time only enters through a second derivative). From a), we see that transforming $t \to t + \frac{\phi}{\omega_0}$, $T \to T$ (T is much slower), we get,

$$u(t,T) \sim A(T)e^{i\omega_0 t}e^{i\phi} + A^*(T)e^{-i\omega_0 t}e^{-i\phi}.$$

If the dynamics of u transformed is to be conserved, then the dynamics of A should be invariant under the transformation $A(T) \to A(T)e^{i\phi}$. This transformation selects the combinations of A, A^*, $\partial_T A$, $\partial_T A^*$ that can appears into $\partial_T^2 A$: only those terms that transform with a factor of $e^{i\phi}$. Considering all the combinations up to third order,

$$A \to Ae^{i\phi};\, A^* \to A^*e^{-i\phi};\, \partial_T A \to \partial_T A e^{i\phi};\, \partial_T A^* \to \partial_T A^* e^{-i\phi};\, A^2 \to A^2 e^{2i\phi};\, AA^* \to AA^*;\, A^{*2} \to A^{*2}e^{-2i\phi}$$
$$A\partial_T A \to A\partial_T A e^{2i\phi};\, A\partial_T A^* \to A\partial_T A^*;\, A^*\partial_T A^* \to A^*\partial_T A^* e^{-2i\phi};\, A^2A^* \to A^2A^*e^{i\phi};\, A^{*2}A \to A^{*2}Ae^{-i\phi};\, \cdots$$

we see that only the terms marked in red can appear in $\partial_T^2 A$, thus,

$$\partial_T^2 A = c_1 A + c_2 \partial_T A + c_3 A^2 A^*,$$

Time-reversal symmetry implies that $t \to -t$, $T \to -T$, $A \to A^*$, which means that c_1 and c_3 are real while c_2 is imaginary. Letting $c_1 = \mu$, $c_2 = i\nu$ and $c_3 = \gamma$ with $\mu, \nu, \gamma \in \mathbb{R}$ we find the final solution,

$$\partial_T^2 A = \mu A + i\nu \partial_T A + \gamma A^2 A^*,$$

7. Show that the second term in Eq. 2 can be removed with a simple change of variables.

   To get ride of the second term, we define $B = Ae^{icT}$, giving,

$$\begin{aligned} \partial_T A &= \partial_T B e^{icT} + icBe^{icT} \\ \partial_T^2 A &= \partial_T^2 B e^{icT} + 2ic\partial_T B e^{icT} - c^2 B e^{icT}, \end{aligned}$$

   which implies,

$$\partial_T^2 B + 2ic\partial_T B - c^2 B = \mu B + i\nu \partial_T B - c\nu B e^{icT} + \gamma B^2 B^*,$$

   Writing $c = \nu/2$ we cancel out the imaginary terms, giving,

$$\partial_T^2 B = \left(\mu - \frac{\nu^2}{4}\right) B + \gamma |B|^2 B.$$

8. Give the condition on γ for the existence of finite amplitude stationary solutions for $\mu > 0$.

   For stationary solutions we need $\partial_T^2 B = 0$, so $\gamma |B|^2 = -\left(\mu - \nu^2/4\right)$.