ICFP M1 - Dynamical Systems and Chaos - TD n°5 - Exercises
Parametric oscillators

Baptiste Coquinot, Stephan Fauve
baptiste.coquinot@ens.fr

2023-2024

1 Inhibition of oscillations by an external forcing. Consider the van der Pol oscillator close to the instability boundary and with an added external forcing such that,

$$\ddot{x} + \omega_0^2 x + \epsilon(x^2 - \mu)\dot{x} = f \sin \Omega t, \tag{1}$$

where $\omega_0 = \mathcal{O}(1)$, $\mu = \mathcal{O}(1)$ and $0 < \epsilon \ll 1$.

1. We first assume that $f = \mathcal{O}(1)$ and $\Omega \neq \omega_0$ and we look for an approximate solution of the form,

$$x(t) = y(t, T) = y_0(t, T) + \epsilon y_1(t, T) + \cdots ,$$

with $T = \epsilon t$. Give the equation for y_0 and show that the solution has two frequencies.

2. Give the governing equation for y_1. Using the solvability condition, find the governing equation for the amplitude of the oscillation at pulsation ω_0.

3. Plot the oscillation amplitude as a function of μ for different values of the forcing f and show that forcing inhibits the oscillation at frequency ω_0.

4. Let us now consider small amplitude forcing with $\Omega = \omega_0 + \epsilon\sigma$, $f = \epsilon F$. Show that in this case, the amplitude of the oscillations behaves as

$$\partial_T A = \frac{\mu}{2} A - \frac{A|A|^2}{2} - \frac{F e^{i\sigma T}}{4\omega_0}.$$

5. Writing the amplitude equation in the frame of reference of the external oscillator, discuss the emergent "resonant forcing" at leading order. In what conditions do we get a quasi-periodic regime?

6. Consider now a small but nearly resonant forcing of a modified van der Pol oscillator with an additional nonlinear term,

$$\ddot{x} + \omega_0^2 x - \epsilon x^2 + \epsilon^2(x^2 - \mu)\dot{x} = \epsilon f \sin \Omega t,$$

where $\Omega = 2\omega_0 + \epsilon^2\nu$, $\nu = \mathcal{O}(1)$. Defining a long time scale $T = \epsilon^2 t$, give the governing equation for $y(t, T) = x(t)$ at $\mathcal{O}(\epsilon^2)$ as a function of Ω, ν, μ and f.

7. Expanding y as $y(t, T) = y_0(t, T) + \epsilon y_1(t, T) + \epsilon^2 y_2(t, T) + \cdots$, find $y_0(t, T)$ and $y_1(t, T)$.

8. Use the solvability condition at next order to find the governing equation for the complex amplitude of the oscillation at pulsation $\Omega/2$. Interpret the result.

2 Parametric excitation The Mathieu equation describes the small amplitude oscillations of a pendulum whose length changes slightly in time with the same frequency as the natural oscillations ($k = \mathcal{O}(1)$),

$$\ddot{x} + (1 + k\epsilon^2 + \epsilon \cos t)x = 0. \tag{2}$$

1. Introduce a slow time scale $T = \epsilon^2 t$ and write down the governing equation for $y(t, T) = x$ in terms of

2. Expanding as $y(t, T) = y_0(t, T) + \epsilon y_1(t, T) + \epsilon^2 y_2(t, T) + \mathcal{O}(\epsilon^3)$, find an expression for $y_0(t, T)$

3. Give the solution y_1 at order ϵ.

4. Use the solvability condition at next order to find the governing equation for the complex amplitude of the oscillation. Under which conditions do the oscillations grow?