ICFP M1 - Dynamical Systems and Chaos - TD n°5 - Solutions
Parametric oscillators

Baptiste Coquinot, Stephan Fauve
baptiste.coquinot@ens.fr

2023-2024

1 Inhibition of oscillations by an external forcing. Consider the van der Pol oscillator close to the instability boundary and with an added external forcing such that,

$$\ddot{x} + \omega_0^2 x + \epsilon(x^2 - \mu)\dot{x} = f \sin \Omega t, \tag{1}$$

where $\omega_0 = \mathcal{O}(1)$, $\mu = \mathcal{O}(1)$ and $0 < \epsilon \ll 1$.

1. We first assume that $f = \mathcal{O}(1)$ and $\Omega \neq \omega_0$ and we look for an approximate solution of the form,

   $$x(t) = y(t, T) = y_0(t, T) + \epsilon y_1(t, T) + \cdots ,$$

   with $T = \epsilon t$. Give the equation for y_0 and show that the solution has two frequencies.

   The derivatives transform as,

   $$\frac{d}{dt} = \partial_t + \epsilon \partial_T$$
   $$\frac{d^2}{dt^2} = \partial_t^2 + 2\epsilon\partial_{Tt} + \mathcal{O}(\epsilon^2),$$

   and Eq. 1 transforms to,

   $$\partial_t^2 y + 2\epsilon\partial_{tT} y + \omega_0^2 y + \epsilon(y^2 - \mu)\partial_t y = f \sin \Omega t + \mathcal{O}(\epsilon^2).$$

   Plugging in $y(t, T) = y_0(t, T) + \epsilon y_1(t, T)$ we find,

   $$\partial_t^2 y_0 + \omega_0^2 y_0 + \epsilon \left[\partial_t^2 y_1 + 2\partial_{tT} y_0 + \omega_0^2 y_1 + \partial_t y_0 (y_0^2 - \mu)\right] = f \sin \Omega t,$$

   which at $\mathcal{O}(1)$ gives,

   $$\partial_t^2 y_0 + \omega_0^2 y_0 = f \sin \Omega t.$$

   The homogeneous part of the solution yields $y_0^h = Ae^{i\omega_0 t}$ + c.c, and the inhomogeneous part should behave as $y_0^i = Af \sin \Omega t + Bf \cos \Omega t$. Plugging this ansatz into the previous equation we get $B = 0$ and $A = 1/(\omega_0^2 - \Omega^2)$, so the general solution is,

   $$y_0(t) = y_0^h(t) + y_0^i(t) = Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t} + \frac{f \sin \Omega t}{\omega_0^2 - \Omega^2}.$$

   Indeed, the solution has two distinct oscillation frequencies, ω_0 and Ω coming from the homogeneous and inhomogeneous terms respectively.

2. Give the governing equation for y_1. Using the solvability condition, find the governing equation for the amplitude of the oscillation at pulsation ω_0.

   At $O(\epsilon)$ we have,

   $$\partial_t^2 y_1 + \omega_0^2 y_1 = (\mu - y_0^2)\partial_t y_0 - 2\partial_{Tt} y_0.$$

   Plugging the solution to y_0 and after some simplification steps we find that at pulsation ω_0 we have

   $$\partial_t^2 y_1 + \omega_0^2 y_1 = -e^{i\omega_0 t}\left[i\omega_0 A^* A^2 + i\omega_0 A \left(\frac{1}{2}\frac{f^2}{(\omega_0^2 - \Omega^2)^2} - \mu\right) + 2i\omega_0 \partial_T A\right] + \text{c.c.} + \text{non-resonant terms.}$$

   Getting rid of secular terms we find,

   $$\partial_T A = \left(\frac{\mu}{2} - \frac{f^2}{4(\omega_0^2 - \Omega^2)^2}\right) A - \frac{1}{2}|A|^2 A$$

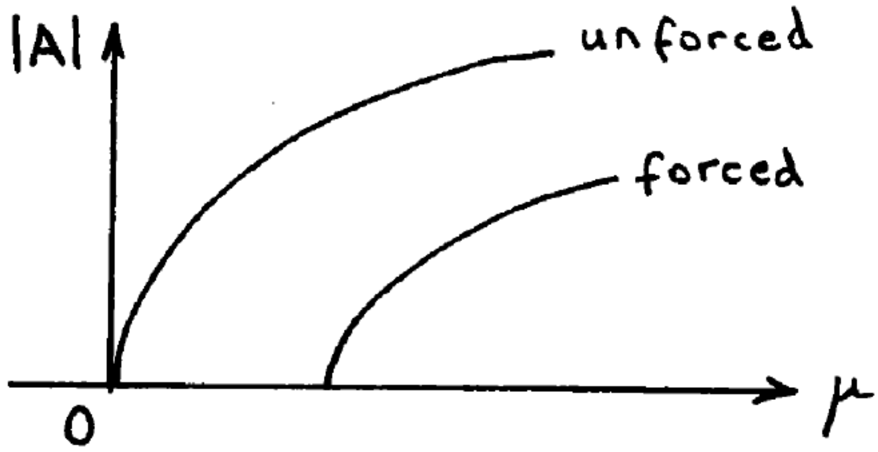

Figure 1: Oscillation amplitude as a function of μ. The forcing inhibits the frequency of oscillations.

3. Plot the oscillation amplitude as a function of μ for different values of the forcing f and show that forcing inhibits the oscillation at frequency ω_0.

   The stationary solution $\partial_T A = 0$ reads,

$$|A|^2 = \mu - \frac{f^2}{2(\omega_0^2 - \Omega_0^2)^2}.$$

   So the forcing inhibits the oscillation amplitudes, Fig. 1.

4. Let us now consider small amplitude forcing with $\Omega = \omega_0 + \epsilon\sigma$, $f = \epsilon F$. Show that in this case, the amplitude of the oscillations behaves as

$$\partial_T A = \frac{\mu}{2}A - \frac{A|A|^2}{2} - \frac{Fe^{i\sigma T}}{4\omega_0}.$$

   Writing $\mathcal{L} = \partial_t^2 + \omega_0^2$, in this case we get,

$$\mathcal{L}y_0 + \epsilon\left[\mathcal{L}y_1 + 2\partial_{tT}y_0 + \partial_t y_0(y_0^2 - \mu)\right] = \epsilon F\sin(\omega_0 t + \epsilon\sigma t).$$

   So in this case at $\mathcal{O}(1)$ we have

$$\mathcal{L}y_0 = 0$$
$$y_0 = Ae^{i\omega_0 t} + \overline{A}e^{-i\omega_0 t}$$

   and the forcing appears at $\mathcal{O}(\epsilon)$,

$$\mathcal{L}y_1 + 2\partial_T Ai\omega_0 e^{i\omega_0 t} + c.c. + \left(i\omega_0 Ae^{i\omega_0 t} + c.c.\right)\left(A^2e^{2i\omega_0 t} + c.c + 2|A|^2 - \mu\right) = F\frac{e^{i\omega_0 t}e^{i\sigma\epsilon t} - e^{-i\omega_0 t}e^{-i\sigma\epsilon t}}{2i}$$

   Collecting the terms in $e^{i\omega_0 t}$ we find and noting that $\epsilon t = T$,

$$\partial_T A = \frac{\mu}{2}A - \frac{|A|^2 A}{2} - \frac{Fe^{i\sigma T}}{4\omega_0}.$$

5. Writing the amplitude equation in the frame of reference of the external oscillator, discuss the emergent "resonant forcing" at leading order. In what conditions do we get a quasi-periodic regime?

   Placing the equation in the frame of reference of the external oscillator can be achieved by applying a coordinate transformation $A = Be^{i\sigma t}$ which yields,

$$\partial_T B = (\mu - i\sigma)B - \frac{|B|^2 B}{2} - \frac{F}{4\omega_0}$$

   This transformation amounts to writing $y_0 = B(T)e^{i\Omega t} + c.c.$ and thus we look at the amplitude equation in the reference frame of the forcing oscillator. That's the sense in which we get a resonant forcing, with an amplitude that grows with the strength of the forcing. However, the fact that we are not forcing at the resonant frequency ($\sigma \neq 0$) results in a detuning that gives rise to a quasi-periodic regime.

6. Consider now a small but nearly resonant forcing of a modified van der Pol oscillator with an additional nonlinear term,

$$\ddot{x} + \omega_0^2 x - \epsilon x^2 + \epsilon^2(x^2 - \mu)\dot{x} = \epsilon f\sin\Omega t,$$

where $\Omega = 2\omega_0 + \epsilon^2\nu$, $\nu = \mathcal{O}(1)$. Defining a long time scale $T = \epsilon^2 t$, give the governing equation for $y(t,T) = x(t)$ at $\mathcal{O}(\epsilon^2)$ as a function of Ω, ν, μ and f.

The derivatives transform as $\frac{d}{dt} = \partial_t + \epsilon^2\partial_T$ and $\frac{d^2}{dt^2} = \partial_t^2 + 2\epsilon^2\partial_{tT}$. Rewriting $\omega_0^2 = (\Omega/2)^2 - \Omega\epsilon^2\nu/2$ we find,

$$\partial_t^2 y + 2\epsilon^2\partial_{tT}y + \frac{\Omega^2}{4}y - \epsilon^2\frac{\nu\Omega}{2}y - \epsilon y^2 + \epsilon^2(y^2-\mu)\partial_t y = \epsilon f\sin(\Omega t).$$

7. Expanding y as $y(t,T) = y_0(t,T) + \epsilon y_1(t,T) + \epsilon^2 y_2(t,T) + \cdots$, find $y_0(t,T)$ and $y_1(t,T)$.

With $y(t,T) = y_0(t,T) + \epsilon y_1(t,T) + \epsilon^2 y_2(t,T)$ we have $y^2 = y_0^2 + 2\epsilon y_0 y_1 + cdots$, giving

$$\partial_t^2 y_0 + \epsilon\partial_t^2 y_1 + \epsilon^2\partial_t^2 y_2 + 2\epsilon^2\partial_{tT}y_0 + \frac{\Omega^2}{4}y_0 +$$
$$\epsilon\frac{\Omega^2}{4}y_1 + \epsilon^2\frac{\Omega^2}{4}y_2 - \epsilon^2\frac{\nu\Omega}{2}y_0 - \epsilon y_0^2 - 2\epsilon^2 y_0 y_1 + \epsilon^2(y_0^2-\mu)\partial_t y_0 = \epsilon f\sin(\Omega t).$$

Therefore, at $\mathcal{O}(1)$ we have

$$\mathcal{L}y_0 = 0$$
$$y_0(t,T) = A(T)e^{i\frac{\Omega}{2}t} + c.c.$$

Plugging this into the solution at $\mathcal{O}(\epsilon)$ we find,

$$\partial_t^2 y_1 + \frac{\Omega^2}{4}y_1 - (A^2e^{i\Omega t} + c.c. + 2|A|^2) = f\sin(\Omega t).$$
$$\mathcal{L}y_1 = A^2e^{i\Omega t} + c.c. + 2|A|^2 + f\sin(\Omega t).$$

The homogeneous part of the equation $\mathcal{L}y_1 = 0$ gives $y_1^h = Be^{i\frac{\Omega}{2}t} + c.c.$, whereas the inhomogeneous part gives

$$y_1^{inh} = -\frac{4}{3\Omega^2}\left(A^2e^{i\Omega t} + c.c - 6|A|^2 + f\sin\Omega t\right)$$

yielding,

$$y_1(t,T) = Be^{i\frac{\Omega}{2}t} + c.c. - \frac{4}{3\Omega^2}\left(A^2e^{i\Omega t} + c.c - 6|A|^2 + f\sin\Omega t\right)$$

8. Use the solvability condition at next order to find the governing equation for the complex amplitude of the oscillation at pulsation $\Omega/2$. Interpret the result.

At $\mathcal{O}(\epsilon^2)$ we have.

$$\partial_t^2 y_2 + 2\partial_{tT}y_0 + \frac{\Omega^2}{4}y_2 - \frac{\nu\Omega}{2}y_0 - 2y_0y_1 + (y_0^2-\mu)\partial_t y_0 = 0,$$

from which we need to collect the terms in $e^{i\frac{\Omega}{2}t}$. From $\partial_{tT}y_0$ we get $i\frac{\Omega}{2}\partial_T A$. Then, from y_0 we get $A(T)$, from y_0y_1 we get,

$$\left(Ae^{i\frac{\Omega}{2}t} + \overline{A}e^{-i\frac{\Omega}{2}t}\right)\left[-\frac{4}{3\Omega^2}\left(A^2e^{i\Omega t} + \overline{A}^2e^{-i\Omega t} - 6|A|^2 + f\frac{e^{i\Omega t}-e^{-i\Omega t}}{2i}\right)\right]$$
$$= \frac{20}{3\Omega^2}|A|^2A + \frac{2if}{3\Omega^2}\overline{A}.$$

Finally, from $(y_0^2-\mu)\partial_t y_0$ we get,

$$\left(A^2e^{i\Omega t} + \overline{A}^2e^{-i\Omega t} + 2|A|^2 - \mu\right)\left(i\frac{\Omega}{2}Ae^{i\frac{\Omega}{2}t} - i\frac{\Omega}{2}\overline{A}e^{-i\frac{\Omega}{2}t}\right)$$
$$= i\frac{\Omega}{2}|A|^2A - i\mu\frac{\Omega}{2}A$$

Putting everything together we find,

$$i\Omega\partial_T A - \frac{\nu\Omega}{2}A - 2\left(\frac{20}{3\Omega^2}|A|^2A + \frac{2if}{3\Omega^2}\overline{A}\right) + i\Omega|A|^2A - i\mu\frac{\Omega}{2}A = 0$$
$$i\Omega\partial_T A - \frac{\Omega}{2}A(\nu + i\mu) - \frac{4if}{3\Omega^2}\overline{A} + \left(\frac{i\Omega}{2} - \frac{40}{3\Omega^2}\right)|A|^2A = 0$$
$$i\Omega\partial_T A = \frac{\Omega}{2}A(\nu + i\mu) + \frac{4if}{3\Omega^2}\overline{A} + \left(\frac{40}{3\Omega^2} - \frac{i\Omega}{2}\right)|A|^2A$$
$$\partial_T A = \frac{1}{2}A(\mu - i\nu) + \frac{4f}{3\Omega^3}\overline{A} - \left(\frac{1}{2} - \frac{40i}{3\Omega^3}\right)|A|^2A$$

2 Parametric excitation The Mathieu equation describes the small amplitude oscillations of a pendulum whose length changes slightly in time with the same frequency as the natural oscillations ($k = \mathcal{O}(1)$),

$$\ddot{x} + (1 + k\epsilon^2 + \epsilon \cos t)x = 0. \tag{2}$$

1. Introduce a slow time scale $T = \epsilon^2 t$ and write down the governing equation for $y(t, T) = x$

   The derivatives transform as $\frac{d}{dt} = \partial_t + \epsilon^2 \partial_T$ and $\frac{d^2}{dt^2} = \partial_t^2 + 2\epsilon^2 \partial_{tT}$, thus giving,

$$\partial_t^2 y + (1 + k\epsilon^2 + \epsilon \cos t)y + 2\epsilon^2 \partial_{tT} y = 0$$

2. Expanding as $y(t, T) = y_0(t, T) + \epsilon y_1(t, T) + \epsilon^2 y_2(t, T) + \mathcal{O}(\epsilon^3)$, find an expression for $y_0(t, T)$ in terms of trignometric functions.

   At $\mathcal{O}(1)$ we have

$$\partial_t^2 y_0 + \omega_0^2 y_0 = 0$$
$$y_0 = A \cos t + B \sin t.$$

3. Give the solution y_1 at order ϵ.

   At $\mathcal{O}(\epsilon)$ we have,

$$\partial_t^2 y_1 + y_1 + \cos t y_0 = 0,$$

   which solves to,

$$y_1 = C(T) \cos t + D(T) \sin t - \frac{A(T)}{2} + \frac{A(T)}{6} \cos 2t + \frac{B(T)}{6} \sin 2t.$$

4. Use the solvability condition at next order to find the governing equation for the complex amplitude of the oscillation. Under which conditions do the oscillations grow?

   At order ϵ^2 we have,

$$\partial_t^2 y_2 + y_2 + \cos t y_1 + k y_0 + 2\partial_{tT} y_0 = 0.$$

   Noting that $\cos 2t \cos t = \frac{1}{2}(\cos t + \cos 3t)$ and $\sin 2t \cos t = \frac{1}{2}(\sin t + \sin 3t)$ we can collecting the terms in $\cos t$ and $\sin t$ and use the solvability condition to find

$$\partial_T A = \frac{B}{2}\left(k + \frac{1}{12}\right)$$
$$\partial_T B = \frac{A}{2}\left(k - \frac{5}{12}\right).$$

   The solution to this pair of equations will thus grow or decay depending on the sign of $\left(k - \frac{5}{12}\right)\left(k + \frac{1}{12}\right)$. The oscillator is thus unstable (oscillations grow) when $-1/12 < k < 5/12$.