ICFP M1 - Dynamical Systems and Chaos - TD n°6 - Exercises
Chaos: theoretical analysis

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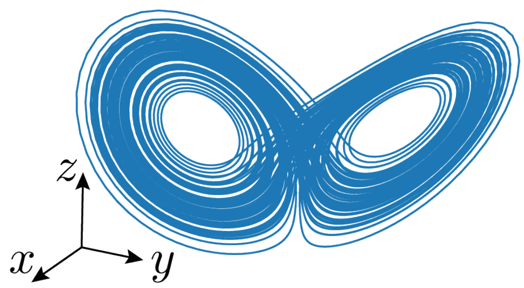

Figure 1: Simulation of the Lorenz sysem with $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$.

1 Lorenz system Consider the Lorenz equations, a simplified model of convection rolls in the atmosphere,

$$\begin{cases} \dot{x} = & \sigma(y-x) \\ \dot{y} = & x(\rho - z) - y \\ \dot{z} = & xy - \beta z \end{cases} . \tag{1}$$

1. Find its fixed points.
2. Find the governing equation for the phase space volume. Under what conditions is the system dissipative?
3. Assuming $\rho, \beta, \sigma > 0$ study the linear stability of the origin with ρ.
4. Find the characteristic equation for the eigenvalues of the Jacobian matrix at the other two fixed points.
5. Seeking solutions of the form $\lambda = i\omega$, where $\omega \in \mathbb{R}$ show that there is a pair of pure imaginary eigenvalues when $\rho = \sigma\left(\frac{\sigma+\beta+3}{\sigma-\beta-1}\right)$.
6. Assume now that the roots of the characteristic equation at $C^{\pm}$ for $\rho > 1$ yield two complex conjugate eigenvalues and a real eigenvalue: $\lambda_{1,2} = a \pm ib$, $\lambda_3 = c$, with $a, b, c \in \mathbb{R}$. Keeping σ and β constant, vary ρ to find that that at ρ_H a pair of complex conjugate eigenvalues crosses the imaginary axis. Find a relationship between σ and β such that the crossing occurs from stable to unstable linear dynamics.
7. Optional: Defining $\epsilon = \rho^{-1/2}$, show that for $\rho \ll 1$ we can rewrite Eq. 1 as,

$$\begin{cases} \dot{u} = & v - \sigma\epsilon u \\ \dot{v} = & -uw - \epsilon v \\ \dot{w} = & uv - \epsilon\beta(w+\sigma) \end{cases} .$$

Find two conserved quantities in the limit $\rho \to \infty$. Is the new system volume preserving for $\rho \to \infty$? Discuss.

2 One-dimensional maps

1. Consider the logistic map $x_{n+1} = \mu x_n(1 - x_n)$ for $0 \leq x_n \leq 1$ and $0 \leq \mu \leq 4$. Find all the fixed points and characterize their stability.
2. Show that for $\mu > 3$ the logistic map has a 2-cycle. (Hint: look for a fixed point of the second-iterate map $f(f(x)) = x$).
3. Show that the 2-cycle is stable for $3 < \mu < 1 + \sqrt{6}$. (Hint: reduce the problem to a question about the stability of a fixed point).
4. When $\mu = 4$ show that $x_n = \sin^2(2^n\theta\pi)$ solves the logistic equation. Find θ in terms of the initial condition. Discuss how this solution highlights two key features of chaos: stretching and folding.

Important: Next week we study chaos numerically. You need to bring this tutorial and your computer.