ICFP M1 - Dynamical Systems and Chaos - TD n°6 - Solutions
Chaos: theoretical analysis

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1 Lorenz system Consider the Lorenz equations, a simplified model of convection rolls in the atmosphere,

$$\begin{cases} \dot{x} = & \sigma(y-x) \\ \dot{y} = & x(\rho - z) - y \\ \dot{z} = & xy - \beta z \end{cases}. \tag{1}$$

1. Find its fixed points.

   The fixed points can be found by taking $(\dot{x}, \dot{y}, \dot{z}) = (0,0,0)$, yielding $(x^*, y^*, z^*) = (0,0,0)$ and $(x^*, y^*, z^*) = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$ (the latter were named C^+ and C^- by Lorenz[1]).

2. Find the governing equation for the phase space volume. Under what conditions is the system dissipative?

   Consider an arbitrary close surface $S(t)$ of volume $V(t)$. In a time dt a patch of area dA sweeps out a volume $\vec{f} \cdot \hat{n} dA$, where $\hat{n}$ is the unit vector normal to S and $\vec{f}$ is the instantaneous veclocity of the points on the surface S. Therefore change in volume is given by $\dot{V} = \int_s \vec{f} \cdot \hat{n} dA$, which is equal to $\dot{V} = \int_V \nabla \vec{f} dV$. For the Lorenz system we thus find,

   $$\dot{V} = (-\sigma - 1 - \beta)V \Rightarrow V(t) = e^{-(\sigma+1+\beta)t} V(0).$$

   This means that the system is dissipative (volumes shrink) whenever $\sigma + 1 + \beta > 0$.

3. Assuming $\rho, \beta, \sigma > 0$ study the linear stability of the origin with ρ.

   To study the linear stability of the origin, we evaluate the Jacobian matrix at $(\dot{x}, \dot{y}, \dot{z}) = (0,0,0)$, yielding,

   $$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

   The dynamics of z is decoupled from the rest and has eigenvalue $-\beta$: $z(t) \to 0$ exponentially fast. The remaining eigenvalues can be obtained by solving the reduced problem

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ \rho & -1 \end{pmatrix}$$

   which has the following eigenvalues

   $$\begin{aligned} (\sigma + \lambda)(1 + \lambda) - \sigma\rho = 0 \\ \lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho) = 0 \\ \lambda = -\frac{\sigma+1}{2} \pm \frac{\sqrt{(\sigma+1)^2 - 4\sigma(1-\rho)}}{2} \end{aligned}$$

   Thus, with $0 < \rho < 1$ all the eigenvalues are negative, and the origin is a stable fixed point. For $\rho > 1$ the origin becomes a saddle, with one positive and one negative exponent. In fact, since $(\sigma+1)^2 - 4\sigma(1-\rho) = \sigma^2 - 2\sigma + 1 + 4\sigma\rho = (\sigma-1)^2 + 4\sigma\rho > 0, \forall \sigma, \rho > 0$, the original has 3 real eigenvalues, with one becoming unstable when $\rho > 1$. Notice that this is a new type of saddle, since the system is 3-dimensional and thus we have two incoming directions and one outgoing direction.

4. Find the characteristic equation for the eigenvalues of the Jacobian matrix at the other two fixed points.

   At C^+ and C^- we have,

   $$J - \lambda\mathbb{1} = \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & \pm\sqrt{\beta(\rho-1)} & -\beta - \lambda \end{pmatrix}.$$

   Taking the determinant we get $\lambda^3 + (\sigma+\beta+1)\lambda^2 + (\rho+\sigma)\beta\lambda + 2\beta\sigma(\rho-1) = 0$.

[1] Lorenz (1963) *Deterministic Nonperiodic Flow.* Journal of the Atmospheric Sciences.

5. Seeking solutions of the form $\lambda = i\omega$, where $\omega \in \mathbb{R}$ show that there is a pair of pure imaginary eigenvalues when $\rho = \sigma\left(\frac{\sigma+\beta+3}{\sigma-\beta-1}\right)$.

   Plugging in $\lambda = i\omega$, we get

$$-i\omega^3 - (\sigma+\beta+1)\omega^2 + i(\rho+\sigma)\beta\omega + 2\beta\sigma(\rho-1) = 0$$

   Taking the real and imaginary parts to zero we findm

$$\begin{aligned}\mathrm{Re}(\cdot) = 0 \Rightarrow & \omega^2 = -\frac{2\beta\sigma(\rho-1)}{\sigma+\beta+1} \\ \mathrm{Im}(\cdot) = 0 \Rightarrow & \omega^3 = -(\rho+\sigma)\beta\omega \Rightarrow \omega^2 = -(\rho+\sigma)\beta\end{aligned}$$

   Equating the real and imaginary parts we find,

$$\rho_H = \sigma\frac{\sigma+\beta+3}{\sigma-\beta-1}.$$

6. Assume now that the roots of the characteristic equation at $C^{\pm}$ for $\rho > 1$ yield two complex conjugate eigenvalues and a real eigenvalue: $\lambda_{1,2} = a \pm ib$, $\lambda_3 = c$, with $a, b, c \in \mathbb{R}$. Keeping σ and β constant, vary ρ to find that that at ρ_H a pair of complex conjugate eigenvalues crosses the imaginary axis. Find a relationship between σ and β such that the crossing occurs from stable to unstable linear dynamics.

   With $\lambda_{1,2} = a \pm ib$ and $\lambda_3 = c$, we have

$$\begin{aligned}(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) &= 0 \\ \lambda^3 - (2a+c)\lambda^2 + (|\lambda_{1,2}|^2 + 2ac)\lambda - |\lambda_{1,2}|^2 c &= 0.\end{aligned}$$

   Equating coefficients with the same powers of λ we find,

$$\begin{aligned}\sigma+\beta+1 &= -(2a+c) \\ \beta(\rho+\sigma) &= |\lambda_{1,2}|^2 + 2ac \\ 2\beta\sigma(\rho-1) &= -|\lambda_{1,2}|^2 c.\end{aligned}$$

   Thus, $c = -(\sigma+\beta+1+2\alpha)$, $(\rho+\sigma)\beta c = 2ac^2 - 2\beta\sigma(\rho-1)$, such that $-(\sigma+\beta+1+2a)(\rho+\sigma)\beta = 2a(\sigma+\beta+1+2c)^2 - 2\sigma\beta(\rho-1)$. Differentiating with respect ρ, then setting $\rho = \rho_H \Rightarrow \alpha(\rho_H) = 0$, we get,

$$\partial_\rho \alpha|_{\rho=\rho_H} = \frac{\beta(\sigma-\beta-1)}{2\left[\beta(\rho_H+\sigma) + (\sigma+\beta+1)^2\right]} > 0, \, \forall \sigma > \beta+1.$$

   Therefore, the eigenvalues cross the imaginary axis with a nonzero positive speed when $\sigma > \beta+1$.

7. Optional: Defining $\epsilon = \rho^{-1/2}$, show that for $\rho \ll 1$ we can rewrite Eq. 1 as,

$$\begin{cases}\dot{u} = & v - \sigma\epsilon u \\ \dot{v} = & -uw - \epsilon v \\ \dot{w} = & uv - \epsilon\beta(w+\sigma)\end{cases}.$$

   Find two conserved quantities in the limit $\rho \to \infty$. Is the new system volume preserving for $\rho \to \infty$? Discuss.

   Scaling (x, y, z) as $x = u/\epsilon$, $y = v/(\epsilon^2\sigma)$ and $z = (w/\sigma + 1)/\epsilon^2$ and defining $t = \epsilon\tau$ we find,

$$\begin{cases}\dot{u} = & v - \sigma\epsilon u \\ \dot{v} = & -uw - \epsilon v \\ \dot{w} = & uv - \epsilon\beta(w+\sigma)\end{cases}.$$

   In the limit $\rho \to \infty$, $\epsilon \to 0$ and as such,

$$\begin{cases}\dot{u} = & v \\ \dot{v} = & -uw \\ \dot{w} = & uv\end{cases}.$$

   Using this, we can define two conserved quantities: $\alpha_1 = u^2 - 2w$ and $\alpha_2 = v^2 + w^2$, since $\dot{\alpha_1} = 0$ and $\dot{\alpha_2} = 0$. The existence of this integrals suggests that the general trajectories in the $\rho \to \infty$ are closed orbits. In fact, we see that proceeding as in 1.) we find that $\dot{V} = 0$, so volumes are preserved by the flow indicating conservative dynamics.

2 One-dimensional maps

1. Consider the logistic map $x_{n+1} = \mu x_n(1 - x_n)$ for $0 \leq x_n \leq 1$ and $0 \leq \mu \leq 4$. Find all the fixed points and characterize their stability.

   For a discrete map, fixed points are solutions of $x_{n+1} = f(x_n)$ so we need to find

$$x^* = \mu x^*(1 - x^*)$$
$$\mu x^2 - (\mu - 1)x = 0$$
$$x^* = 0 \bigvee x^* = (1 - \mu^{-1}).$$

   Which means that the second fixed point only exists when $\mu > 1$. To determine the stability of the fixed points of a map, consider the time evolution of a perturbation around a fixed point $x_n = x^* + \delta_n$

$$x^* + \delta_{n+1} = f(x^* + \delta_n) = f(x^*) + \delta_n f'(x^*) + \mathcal{O}(\delta_n^2)$$
$$\delta_{n+1} = f'(x^*)\delta_n = \lambda\delta_n$$

   since $f(x^*) = x^*$. The solution to this linear map is $\delta_n = \lambda^n \delta_0$ and thus if $|\lambda| = |f'(x^*)| < 1$ we have a stable fixed point, and if $|\lambda| = |f'(x^*)| > 1$ we have an unstable fixed point. In the case $|\lambda| = 1$ we have a marginal case, which requires the use of the $\mathcal{O}(\delta_n^2)$ terms to determine the local stability. In the case of the logistic equation we get $f'(x^*) = \mu(1 - 2x^*)$. Therefore, at the origin $x^* = 0$ we have $|\lambda| = \mu$ so the origin is stable when $\mu < 1$ and unstable when $\mu > 1$. The other fixed point has $f'(x^*) = 2 - \mu \Rightarrow |\lambda| = |2 - \mu|$ so it is stable for $1 < \mu < 3$ and unstable for $\mu > 3$.

2. Show that for $\mu > 3$ the logistic map has a 2-cycle. (Hint: look for a fixed point of the second-iterate map $f(f(x)) = x$).

   A 2-cycle exists when $f(f(p)) = p$: we iterate the system twice and return to the same point; p is a fixed point of the second iterate map.

$$f(f(p)) = p$$
$$\mu^2 x(1 - x)\left[1 - \mu x(1 - x)\right] - x = 0,$$

   So we have a 4th order polynomial to solve and find the fixed points. However, notice that the fixed points of the map $x^* = f(x^*)$ are also fixed points of $x^* = f(f(x^*))$, and therefore we already have two of the roots of the equation. To get the other two we can factor these out to find,

$$f(f(p)) = p$$
$$x(\mu x - \mu + 1)\left[-1 - \mu + \mu^2 x + \mu x - \mu^2 x^2\right] = 0,$$

   and thus we only need to find the roots of the quadratic equation,

$$-1 - \mu + \mu^2 x + \mu x - \mu^2 x^2 = 0$$
$$x = \frac{\mu(\mu + 1) \pm \sqrt{\mu^2(\mu + 1)^2 - 4\mu^2(1 + \mu)}}{2\mu^2}$$
$$x = \frac{\mu + 1 \pm \sqrt{(\mu + 1)^2 - 4\mu^2(1 + \mu)}}{2\mu}$$
$$x = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu},$$

   which is real for $\mu > 3$. Thus a two-cycle exists for $\mu > 3$.

3. Show that the 2-cycle is stable for $3 < \mu < 1 + \sqrt{6}$. (Hint: reduce the problem to a question about the stability of a fixed point).

   To study the stability of the two cycle, we reduce the problem to the stability of a fixed point $x = f(f(x))$. Denoting the 2-cycle fixed points as p and q we have

$$\lambda = \frac{d}{dx}\left[f(f(x))\right]_{x^*=p} = f'(f(p))f'(p) = f'(q)f'(p)$$

   The same is true for the other fixed point due to the symmetry of the final term; this means that when p bifurcates so does q, and vice-versa. So we get,

$$\lambda = \mu^2(1 - 2q)(1 - 2q)$$
$$= \mu^2\left[1 - 2(p + q) + 4pq\right]$$
$$= \mu^2\left[1 - 2(\mu + 1)/\mu + 4(\mu + 1)/\mu^2\right]$$
$$\lambda = 4 + 2\mu - \mu^2.$$

   Therefore, the 2-cycle is stable for $|4 + 2\mu - \mu^2| < 1$, i.e for $3 < \mu < 1 + \sqrt{6}$

4. When $\mu = 4$ show that $x_n = \sin^2(2^n\theta\pi)$ solves the logistic equation. Find θ in terms of the initial condition. Discuss how this solution highlights two key features of chaos: stretching and folding.

Starting from $x_0 = \sin^2(\theta\pi)$ we get,

$$\begin{aligned} x_1 &= 4\sin^2(\theta\pi)\,(1-\sin^2(\theta\pi)) \\ &= 4\sin^2(\theta\pi)\cos^2(\theta\pi) \\ &= \frac{1}{2}(1-\cos^2(4\theta\pi)) = \sin^2(2\theta\pi), \end{aligned}$$

Similarly from x_1,

$$\begin{aligned} x_2 &= 4\sin^2(2\theta\pi)\,(1-\sin^2(2\theta\pi)) \\ &= 4\sin^2(2\theta\pi)\cos^2(2\theta\pi) \\ &= \frac{1}{2}(1-\cos^2(8\theta\pi)) = \sin^2(4\theta\pi), \end{aligned}$$

from which we can deduce that $x_n = \sin^2(2^n\theta\pi)$ solves the logistic equation, with $\theta = \frac{\arcsin(\sqrt{x_0})}{\pi}$. You can also deriving it more generally by making use of more complicated trignometric identities,

$$\begin{aligned} x_{n+1} &= 4\sin^2(2^n\theta\pi)\,(1-\sin^2(2^n\theta\pi)) \\ &= 4\sin^2(2^n\theta\pi)\cos^2(2^n\theta\pi) \\ &= \frac{1}{2}(1-\cos^2(2^{n+2}\theta\pi)) = \sin^2(2^{n+1}\theta\pi), \end{aligned}$$

In some sense, the 2^n factor blows up the term argument of the sin, which is then folded into the $[-1, 1]$ interval by the sin function, highlighting the stretching and folding characteristic of chaos.