ICFP M1 - Dynamical Systems and Chaos - TD n°7 - Exercises
Chaos: numerical analysis

Baptiste Coquinot, Stephan Fauve
baptiste.coquinot@ens.fr

2023-2024

In this TD we will explore numerical methods for the characterization of chaotic dynamical systems. Check https://colab.research.google.com/drive/1Qb5egNHlA7ZIsTyCzqlImjT_yknaE9-B for code and solutions.

1 Lorenz system Consider the Lorenz equations, a simplified model of convection rolls in the atmosphere,

$$\begin{cases} \dot{x} = & \sigma(y-x) \\ \dot{y} = & x(\rho - z) - y \\ \dot{z} = & xy - \beta z \end{cases} . \tag{1}$$

1. Using numerical simulations, fix $\sigma = 10$ and $\beta = 8/3$ and explore the behavior of the Lorenz system for different values of ρ illustrative of the different dynamical regimes discussed in the problem 1.1) of TD5. You can use the `scipy.integrate.odeint` package (simulate for $T = 50\,\mathrm{s}$ with a time step of $\delta t = 0.1\,\mathrm{s}$).

2. Suppose that we slowly turn the ρ knob up and down around $\rho = \rho_H - .2$ (check TD5 for ρ_H) in a sinusoidal fashion $\rho(t) = \rho_H - 0.2 + \sin(\omega t)$ where ω is slow compared to typical orbital frequencies on the attractor. Numerically integrate the equations, and plot the solutions in whatever way seems most revealing. Discuss the results.

3. Exploring the exponential divergence of trajectories in chaotic systems. Simulate the Lorenz system in the standard chaotic regime ($\sigma = 10$, $\beta = 8/3$ and $\rho = 28$) for a large time $T = 100\,\mathrm{s}$ with $\delta t = 0.01\,\mathrm{s}$. Pick a point in the attractor at random $\vec{x}_0$, sample a point $\hat{\vec{x}}$ at an initial distance $\delta_0 = 10^{-5}$ from it, $||\hat{\vec{x}} - \vec{x}_0|| = \delta$, and follow $\delta(t) = ||\hat{\vec{x}}(T) - \vec{x}_0(t)||$. How does this distance evolve in time? Show through numerical experiments that the distances grow initially as $\delta(t) \sim \delta_0 e^{\lambda t}$, where λ is the largest Lyapunov exponent of the dynamics.

4. Estimate the average rate of separation between trajectories by repeating the calculation in 1.3) N times, $\hat{\delta} = \langle \delta_i(t) \rangle = e^{\lambda t}\delta_0$. Estimate the Lyapunov exponent.

5. Compute $\delta(t)$ for the different dynamical regimes discussed in 1.1). Discuss the behavior of the curves for the different dynamical regimes.

6. A positive Lyapunov exponent poses a fundamental upper bound to the predictability horizon of the dynamical system. Starting from an initial error δ_0, how does the timescale to obtain an error Δ depend on the Lyapunov exponent? Make the initial error δ_0 much smaller and estimate how much better we can predict the system?

(optional but cool) Lyapunov spectrum estimation In a d-dimensional state-space there are actually d Lyapunov exponents which measure the rates of separation of nearby trajectories along d orthogonal directions. These directions are determined by the flow. The stretching factors in each of these chosen directions constitute the Lyapunov spectrum of the system. To estimate the Lyapunov spectrum, we evolve a sphere of initial conditions and compute the stretching and contraction of the main axis of the resulting ellipsoid. Given an initial sphere B_0 centered at v_0 in $\mathbb{R}^n$, we act on it with a power of the jacobian matrix $J_t = Df^t(v_0)$, yielding an ellipsoid $B^t = B^0 J_t$ whose axis lengths are given by the eigenvalues of $J_t J_t^T$, σ_t, Fig. 1 [1]. In practice, the direct calculation of Lyapunov exponents for large t is plagued by numerical errors: the existence of strongly contracting or expanding directions means that for large t there will be very small and very large eigenvalues. Because of the limited number of digits allowed for each stored number, computer calculations become difficult when numbers of vastly different sizes are involved in the same calculation. The problem of computing the eigenvalues thus gets worse as t increases. For this reason, the direct calculation of the ellipsoid $B_0 J_t$ is usually avoided. Instead, we follow the ellipsoid as it morphs: $J_t U = Df(v_{n-1}) \cdots Df(v_1) Df(v_0) B^0$. Starting from an initial orthonormal basis, we evolve the basis vectors through $Df(v_0)$ and use a Gram-Schmidt orthogonalization through a QR decomposition to find a new orthogonal basis for the ellipsoids in the t time step. Applying this transformation t times we can follow how the Lyapunov exponents evolve. In the google colab there is code for the Lyapunov spectrum estimation, given a python function that evaluate the jacobian at a state-space point s given parameters σ, ρ and β `jacobian(s,sigma,rho,beta)`.

[1] See e.g. Alligood, K.T., Sauer, T., Yorke, J.A., & Crawford, J.D. (1997). Chaos: An Introduction to Dynamical Systems

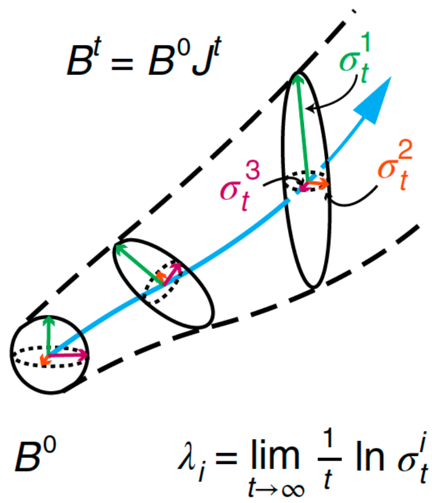

Figure 1: Schematic of the Lyapunov spectrum estimation

7. Write down the function for estimating the jacobian, and estimate the Lyapunov spectrum of the Lorenz system in the standard chaotic regime. Compare this results with 1.4) of this TD and with problem 1.2) of TD5.

2 Lyapunov exponents and entropy: the logistic equation

In this section we will glimpse at the deep connections between entropy and the lyapunov exponents through the Logistic equation $x_{n+1} = \mu x_n(1 - x_n)$.

1. Period doubling and orbit diagrams. For a range of values of μ, generate an orbit starting from some random initial condition x_0 and iterate for 200 steps to allow the system to settle down to its steady-state behavior. Once the transients have decayed, plot many points, e.g. $\{x_{100}, \dots, x_{200}\}$, above each value of μ. Discuss the results. Use the orbit diagram to choose values of μ corresponding to the following dynamical regimes: stable fixed point at origin, stable fixed point elsewhere, 2-cycle, 4-cycle, 8-cycle and chaos.

2. Coarse-graining the logistic map. Binarize the dynamics of the Logistic map by setting $x(t) > .5 \to 1$ and $x(t) \leq .5 \to 0$ and look at the sequence of symbols generated for different illustrative values of μ. How easy is it to predict the next symbol in a sequence depending on the current symbol? How does this depend on the different dynamical regimes μ?

3. The unpredictability of the resulting symbolic sequences can be quantified through the entropy rate, which corresponds to the entropy of the conditional distribution of future states given the current states. A simple way to estimate the entropy rate is to count the number of distinct sequences of length K, S_K^i. We can count how many sequences of a specific kind we observe by constructing $s_K = \hat{x}_t, \hat{x}_{t+1}, \dots, \hat{x}_{t+K}$ (where $\hat{x}$ represents the discretized x) and counting the number of times we observe a given sequence S_K^i to obtain the probability $p(S_K^i)$ of observing the sequence in the dataset. From this, we can estimate the sequence entropy, $H_K = -\sum_i^N p(S_K^i) \log p(S_K^i)$ and then obtain the entropy rate, or entropy per symbol by estimating, $h = \lim_{K\to\infty} H_K/K$. Estimate the entropy rate h for different values of $\mu \to 4$ (going beyond $K = 5$ is computationally very expensive). How does this compare the a coin-tossing experiment? And to 1-dimensional chain Ising spins at high temperature?

4. In a 1-d discrete map $x_{n+1} = f(x_n)$, show that the Lyapunov exponent can be simply estimated by,

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)| \tag{2}$$

5. As you probably suspected, the unpredictability of the dynamics is related to the positive Lyapunov exponents, which capture the degree by which two nearby trajectories diverge in time and therefore how fast the system becomes unpredictable. In fact, the Pesin identity[2] dictates that for systems such as the logistic map, the Kolmogorov-Sinai entropy rate h should correspond to the sum of positive Lyapunov exponents, $h = \sum_{\lambda_i > 0} \lambda_i$ (the overall rate of expansion of phase space volumes). Using numerical simulations, show that there is a correspondence between the Lyapunov exponent and the entropy rate for the Logistic map with $\mu \to 4$.

6. Estimate the Lyapunov exponents in the range $\mu \in [3.4, 4[$ and discuss in light of the results from 2.1).

[2]Pesin, Y. B. (1977). "Characteristic Lyapunov Exponents and Smooth Ergodic Theory"