ICFP M1 - Dynamical Systems and Chaos - TD n°8 - Exercises
Markov Processes

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A random process X_t can be viewed as a family of random numbers, indexed by the label t. For each time t, X_t may obey a different probability distribution $p(x,t)$. The values of the random process at different times t, t' may or may not depend on each other. The conditional probability $p(x_n, t_n|x_{n-1}, t_{n-1}, ...x_1, t_1)$ is defined as the probability of X_{t_n} taking the value x_n, given that X_{t_i} takes the value x_i for each $i \in \{1, ..., n-1\}$. If

$$p(x_n, t_n|x_{n-1}, t_{n-1}; ...; x_1, t_1) = p(x_n, t_n), \tag{1}$$

X_t is a *purely random process*, where the values of X_t at different times are independent, which cannot describe a physical continuous dependence on time. The second simplest case,

$$p(x_n, t_n|x_{n-1}, t_{n-1}; ...; x_1, t_1) = p(x_n, t_n|x_{n-1}, t_{n-1}), \tag{2}$$

defines a *Markov process.* One also calls $p(x, t|x', t')$ *transition probability.*

Basics of Markov Chains.

1. Show that for Markov process, the n-point joint probability density reduces to

$$p(x_n, t_n; ...; x_1, t_1) = p(x_n, t_n|x_{n-1}, t_{n-1})p(x_{n-1}, t_{n-1}|x_{n-2}, t_{n-2})...p(x_2, t_2|x_1, t_1)p(x_1, t_1). \tag{3}$$

2. Show further that this implies

$$p(x_3, t_3|x_1, t_1) = \int p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)\mathrm{d}x_2. \tag{4}$$

   This relation is known as Chapman-Kolmogorov equation.

3. *(Bonus)* For pure Brownian motion, the transition probability is:

$$p(x_2, t_2|x_1, t_1) = \frac{1}{\sqrt{4\pi(t_2 - t_1)}} e^{\frac{-(x_2-x_1)^2}{4(t_2-t_1)}},$$

   meaning that they depend only on the difference in positions and times. Show that such transition probability satisfies the Chapman-Kolmogorov equation.

The Master Equation. Consider the transition probability from some state x'' at time t to another state x at time $t + \Delta t$ for Δt small,

$$p(x, t + \Delta t|x'', t) = (1 - a(x, t)\Delta t)\delta(x - x'') + W(x, x'', t)\Delta t + O(\Delta t^2). \tag{5}$$

Here the term involving $\delta(x - x'')$ is the probability to be at the same point after Δt, while $W(x, x'', t)$ (*the rate function*) is the probability to transition from x'' to x within the time interval Δt.

4. Determine $a(x, t)$ from the constraint of normalisation.

5. Use the Chapman-Kolmogorov equation to show that

$$\partial_t p(x, t|x', t') = \int \left[W(x, x'', t)p(x'', t|x', t') - W(x'', x, t)p(x, t|x', t')\right] \mathrm{d}x''. \tag{6}$$

   This is the so-called *continuous-time master equation*, which implies,

$$\partial_t p(x, t) = \int \left[W(x, x', t)p(x', t) - W(x', x, t)p(x, t)\right] \mathrm{d}x'. \tag{7}$$

The Fokker-Planck Equation. We now want to perform an expansion to find a partial differential equation describing our process.

6. Write $W(x, x', t) = w(x', r, t)$ with $r = x - x'$. Show that the Master equation implies

$$\partial_t p(x,t) = \int [w(x-r,r,t)p(x-r,t) - w(x,-r,t)p(x,t)]\,\mathrm{d}r. \tag{8}$$

Expand the first argument of $w(x-r,r,t)p(x-r,t)$ around x (*Kramers-Moyale expansion*) to show that

$$\partial_t p(x,t) = \sum_{n=1}^{\infty} (-\partial_x)^n\,[D_n(x,t)p(x,t)], \tag{9}$$

where $D_n = \frac{1}{n!}\int w(x,r,t)r^n \mathrm{d}r$. This series may terminate at order 2, in which case we obtain the Fokker-Planck equation:

$$\partial_t p(x,t) = -\partial_x[D_1(x,t)p(x,t)] + \partial_x^2[D_2(x,t)p(x,t)]. \tag{10}$$

7. Show that the Fokker-Planck equation can be written as a conservation law $\partial_t p = \partial_x J$, write down J.

8. Assume $x \in \mathbb{R}$ and $p(x,t) \overset{x\to\pm\infty}{\to} 0$ sufficiently fast. What equation does the mean $\langle x \rangle$ obey?

9. Given two solutions $p_1(x,t), p_2(x,t)$ of the Fokker-Planck equation starting from different initial conditions, consider $H(t) = \int p_1 \ln(p_1/p_2)\mathrm{d}x$, that we assume well defined. Show that $H(t) \geq 0$ and that $\frac{\mathrm{d}}{\mathrm{d}t}H(t) \leq 0$. What does this tell us about the long-time behaviour of the solutions? Discuss.