ICFP M1 - Phase Transitions – TD n° 1
The Curie-Weiss Model

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We consider a system of N Ising spins, with configurations denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, that interact according to the Hamiltonian

$$H(\underline{\sigma}) = -\frac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \ , \tag{1}$$

where $J > 0$. This model is called the Curie-Weiss model. It is a mean-field model because each degree of freedom interacts with all others. In this sense, it does not show any geometric structure.

1. Why is the coupling constant J multiplied by a factor of $1/N$?

2. Suppose the system is in equilibrium with a heat bath of temperature T. Noting that H depends on $\underline{\sigma}$ only through the average magnetization $m(\underline{\sigma}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \in [-1, 1]$ put the partition function $Z = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}$ under the form:

$$Z = \sum_{m \in \mathcal{M}_N} \mathcal{N}_m^N \, e^{-\beta N[-\frac{J}{2} m^2 - hm]} \ . \tag{2}$$

   Specify the set $\mathcal{M}_N$ in which m varies, as well as the value of $\mathcal{N}_m^N$.

3. We define the free energy per spin in the thermodynamic limit as

$$f_{\mathrm{CW}}(\beta, h) = \lim_{N \to \infty} -\frac{1}{\beta N} \ln Z \ . \tag{3}$$

   Show that in the limit $N \to \infty$ the following identity holds:

$$\frac{1}{N} \ln \binom{N}{\alpha N} = -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha) + O\left(\frac{\ln N}{N}\right), \tag{4}$$

   where $\alpha \in [0, 1]$. Use this to deduce

$$f_{\mathrm{CW}}(\beta, h) = \inf_{m \in [-1,1]} \widehat{f}_{\mathrm{CW}}(m; \beta, h) \ , \tag{5}$$

   where you should specify the function $\widehat{f}_{\mathrm{CW}}(m; \beta, h)$.

4. Sketch the behavior of $\widehat{f}_{\mathrm{CW}}(m; \beta, h = 0)$ as a function of m for different temperatures. What is the value of the critical temperature T_c where this behavior changes qualitatively?

5. What is the effect of a nonzero field $h > 0$ on these curves?

6. We define $m_*(\beta, h)$ as the magnetization which minimizes $\widehat{f}_{\mathrm{CW}}(m; \beta, h)$. Give an implicit equation satisfied by $m_*(\beta, h)$.

7. Sketch the behavior of $m_*(\beta, h)$ as a function of h for a temperature larger and a temperature smaller than T_c.

8. We define the spontaneous magnetization as $m_{\rm sp}(\beta) = \lim_{h\to 0^+} m_*(\beta,h)$. Sketch its behavior as a function of the temperature. Determine the exponent $\beta_{\rm mf}$ that characterizes the behavior of $m_{\rm sp}$ in a neighborhood of $T_{\rm c}$, i.e. such that $m_{\rm sp}(T) \propto (T_{\rm c}-T)^{\beta_{\rm mf}}$ when $T \to T_{\rm c}^-$, where $\propto$ denotes an equivalence up to a multiplicative constant.

9. Compute the exponent δ that describes the behavior of the magnetization as a function of the magnetic field at the critical temperature, according to the definition $m_*(\beta_{\rm c},h) \propto h^{1/\delta}$ when $h \to 0^+$.

10. Express the magnetic susceptibility $\chi(\beta,h) = \frac{\partial m_*}{\partial h}$ as a function of $m_*(\beta,h)$. Determine then the exponent γ that controls its divergence at the critical point according to $\chi(\beta,h=0^+) \propto |T-T_{\rm c}|^{-\gamma}$.

11. Express in terms of $m_*(\beta,h)$ the average energy per spin, $u = \lim_{N\to\infty} \frac{1}{N}\langle H\rangle$. How does it behave as a function of the temperature, for $h=0$, in the high-temperature phase ? And when approaching the critical point from the low-temperature phase ? Justify the value $\alpha = 0$ attributed in this model to the critical exponent α describing the behavior of the specific heat according to $C = \frac{\partial u}{\partial T}\big|_{h=0} \propto |T-T_{\rm c}|^{-\alpha}$.

Useful formulas:

- $\mathrm{th}(x) = x - \frac{x^3}{3} + O(x^5)$ for $x \to 0$,
- $\mathrm{th}'(x) = 1 - \mathrm{th}^2(x)$.