

ICFP M1 - PHASE TRANSITIONS – TD n° 1
The Curie-Weiss Model

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We consider a system of N Ising spins, with configurations denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, that interact according to the Hamiltonian

$$H(\underline{\sigma}) = -\frac{J}{2N} \sum_{i,j=1}^N \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i, \quad (1)$$

where $J > 0$. This model is called the Curie-Weiss model. It is a mean-field model because each degree of freedom interacts with all others. In this sense, it does not show any geometric structure.

1. Why is the coupling constant J multiplied by a factor of $1/N$?
2. Suppose the system is in equilibrium with a heat bath of temperature T . Noting that H depends on $\underline{\sigma}$ only through the average magnetization $m(\underline{\sigma}) = \frac{1}{N} \sum_{i=1}^N \sigma_i \in [-1, 1]$ put the partition function $Z = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}$ under the form:

$$Z = \sum_{m \in \mathcal{M}_N} \mathcal{N}_m^N e^{-\beta N[-\frac{J}{2}m^2 - hm]}. \quad (2)$$

Specify the set \mathcal{M}_N in which m varies, as well as the value of \mathcal{N}_m^N .

3. We define the free energy per spin in the thermodynamic limit as

$$f_{\text{CW}}(\beta, h) = \lim_{N \rightarrow \infty} -\frac{1}{\beta N} \ln Z. \quad (3)$$

Show that in the limit $N \rightarrow \infty$ the following identity holds:

$$\frac{1}{N} \ln \binom{N}{\alpha N} = -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha) + O\left(\frac{\ln N}{N}\right), \quad (4)$$

where $\alpha \in [0, 1]$. Use this to deduce

$$f_{\text{CW}}(\beta, h) = \inf_{m \in [-1, 1]} \widehat{f}_{\text{CW}}(m; \beta, h), \quad (5)$$

where you should specify the function $\widehat{f}_{\text{CW}}(m; \beta, h)$.

4. Sketch the behavior of $\widehat{f}_{\text{CW}}(m; \beta, h = 0)$ as a function of m for different temperatures. What is the value of the critical temperature T_c where this behavior changes qualitatively?
5. What is the effect of a nonzero field $h > 0$ on these curves?
6. We define $m_*(\beta, h)$ as the magnetization which minimizes $\widehat{f}_{\text{CW}}(m; \beta, h)$. Give an implicit equation satisfied by $m_*(\beta, h)$.
7. Sketch the behavior of $m_*(\beta, h)$ as a function of h for a temperature larger and a temperature smaller than T_c .

8. We define the spontaneous magnetization as $m_{\text{sp}}(\beta) = \lim_{h \rightarrow 0^+} m_*(\beta, h)$. Sketch its behavior as a function of the temperature. Determine the exponent β_{mf} that characterizes the behavior of m_{sp} in a neighborhood of T_c , i.e. such that $m_{\text{sp}}(T) \propto (T_c - T)^{\beta_{\text{mf}}}$ when $T \rightarrow T_c^-$, where \propto denotes an equivalence up to a multiplicative constant.
9. Compute the exponent δ that describes the behavior of the magnetization as a function of the magnetic field at the critical temperature, according to the definition $m_*(\beta_c, h) \propto h^{1/\delta}$ when $h \rightarrow 0^+$.
10. Express the magnetic susceptibility $\chi(\beta, h) = \frac{\partial m_*}{\partial h}$ as a function of $m_*(\beta, h)$. Determine then the exponent γ that controls its divergence at the critical point according to $\chi(\beta, h = 0^+) \propto |T - T_c|^{-\gamma}$.
11. Express in terms of $m_*(\beta, h)$ the average energy per spin, $u = \lim_{N \rightarrow \infty} \frac{1}{N} \langle H \rangle$. How does it behave as a function of the temperature, for $h = 0$, in the high-temperature phase? And when approaching the critical point from the low-temperature phase? Justify the value $\alpha = 0$ attributed in this model to the critical exponent α describing the behavior of the specific heat according to $C = \left. \frac{\partial u}{\partial T} \right|_{h=0} \propto |T - T_c|^{-\alpha}$.

Useful formulas:

- $\text{th}(x) = x - \frac{x^3}{3} + O(x^5)$ for $x \rightarrow 0$,
- $\text{th}'(x) = 1 - \text{th}^2(x)$.