ICFP M1 - Phase Transitions – TD n° 2
Unidimensional Models

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1 Tranfer matrix for unidimensional models

1.1 Ising chain

We shall consider a unidimensional chain of N Ising spins $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, with periodic boundary conditions $\sigma_{N+1} = \sigma_1$, interacting according to the Hamiltonian

$$H(\underline{\sigma}) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i \ , \tag{1}$$

where $J > 0$ correspond to a ferromagnetic coupling between nearest neighbours, and h is the effect of an exterior magnetic field. The system is in equilibrium with a thermal bath, hence the probability of a configuration $\underline{\sigma}$ is $e^{-\beta H(\underline{\sigma})}/Z$, with Z the partition function normalising the distribution. $\langle \bullet \rangle$ indicates the mean with respect to this probability law.

1. Show that the partition function can be written as

$$Z = \sum_{\underline{\sigma}} \prod_{i=1}^{N} T(\sigma_i, \sigma_{i+1}) \ , \tag{2}$$

   with T symmetric (i.e. $T(\sigma, \sigma') = T(\sigma', \sigma)$). In the following $\mathbb{T}$ denotes the 2×2 matrix such that $\mathbb{T}_{\sigma\sigma'} = T(\sigma, \sigma')$. Write down $\mathbb{T}$.

2. Express Z as a function of $\mathbb{T}$.

3. Find the eigenvalues $\lambda_\pm$ of $\mathbb{T}$ (with the convention $\lambda_+ > \lambda_-$).

4. Express the free energy per spin $f = -\frac{1}{N\beta} \ln Z$ as a function of $\lambda_\pm$, and simplify your result in the thermodynamic limit $N \to \infty$.

5. We are interested in the mean magnetization per spin $m = \langle \sigma_i \rangle$. Express m as a function of $\mathbb{T}$ and of the matrix $\hat{\sigma}$:

$$\hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{3}$$

6. Find another expression of m as a derivative of f, and show that in the thermodynamic limit :

$$m = \frac{\text{sh}(\beta h)}{\sqrt{\text{sh}^2(\beta h) + e^{-4\beta J}}} \ . \tag{4}$$

   Plot it as a function of h for several temperatures. Is there a phase transition in this model ?

7. In this question and in the following we take $h = 0$. We consider the correlation function between spins at distance k in a chain of length N, defined as $C_N(k) = \langle \sigma_i \sigma_{i+k} \rangle$. Express it as a function of $\mathbb{T}$ and $\hat{\sigma}$.

8. Show that in the thermodynamic limit ($N \to \infty$ with k fixed) this correlation function simplify as $C(k) = e^{-k/\xi}$, with ξ the correlation length. Give an explicit expression of ξ, and plot is as a function of the temperature. Can we see a phase transition from this quantity ?

1.2 Ising chain with second neighbours interaction

We now consider a 1D Ising chain spin with first and second neighbours interactions:

$$H(\underline{\sigma}) = -J_1 \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - J_2 \sum_{i=1}^{N} \sigma_i \sigma_{i+2} \ , \tag{5}$$

and we extend the periodic boundary condition to $\sigma_{N+2} = \sigma_2$. We assume N is even.

9. Show that Z can be written under the form :

$$Z = \sum_{\underline{\sigma}} \prod_{i=0}^{(N/2)-1} T((\sigma_{2i+1}, \sigma_{2i+2}), (\sigma_{2i+3}, \sigma_{2i+4})) \ . \tag{6}$$

10. Give a matrix $\mathbb{T}$ of order 4 such that $Z = \operatorname{Tr} \mathbb{T}^{(N/2)}$. You do not need to write it explicitly.

    We will admit the two following results :

    - The Perron-Frobenius theorem asserts that a real matrix M whose elements are all strictly positive admit an eigenvalue λ_0 non degenerated and strictly positive, such that the modulus of all the other eigenvalues is strictly smaller than λ_0.
    - Let $P(\lambda, \beta)$ be a unitary polynomial of the variable λ whose coefficients are analytical functions of β. If λ_0 is a simple root of $P(\cdot, \beta_0)$, then there is a function $\lambda(\beta)$, analytic in a neighbourhood of β_0 with $\lambda(\beta_0) = \lambda_0$, such that $P(\lambda(\beta), \beta) = 0$ for all β in a neighborhood of β_0.

11. Using these two results show that there is no phase transition in the model, and more generally in any unidimensional model with variables that take a finite number of values and with finite interaction range.

2 The correlation functions of the unidimensional Ising model via diagrammatic expansions

We return to the case of nearest neighbor interactions only, without magnetic field, considering the Hamiltonian

$$H(\underline{\sigma}) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \ , \tag{7}$$

with periodic boundary conditions, $\sigma_{N+1} = \sigma_1$. We will recover some of the previous results by an alternative method based on diagrammatic expansions.

1. Show that $e^{x\epsilon} = (\operatorname{ch} x)(1 + \epsilon \operatorname{th} x)$ if $\epsilon = \pm 1$.
2. Deduce from that the following expression of the partition function for the Ising model :

$$Z = (\operatorname{ch} \beta J)^N \sum_{\underline{\sigma}} \prod_{i=1}^{N} [1 + \sigma_i \sigma_{i+1} (\operatorname{th} \beta J)] \ . \tag{8}$$

3. Each term in the expansion of the product can be associated to a diagram, i.e. a subset of the edges of the chain, where one retains only the edges $(i, i+1)$ for which the factor proportional to $\sigma_i \sigma_{i+1}$ is chosen. Which diagrams contribute to the sum over $\underline{\sigma}$?
4. Deduce from these considerations the value of Z, and of the free energy per spin in the thermodynamic limit, $f = -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \ln Z$. Check the consistency of these results with those of the first exercise.

5. What is the average magnetization of a spin, $\langle \sigma_i \rangle$? One can follow two reasoning, one based on symmetry considerations, the other by diagrammatic arguments.
6. Compute the two-point correlation function, $\langle \sigma_i \sigma_j \rangle$, with $1 \leq i < j \leq N$, using the diagrammatic expansion. Simplify its expression in the thermodynamic limit ($N \to \infty$ with i, j fixed), and give explicitly the correlation length ξ. Compare with the result of the first exercise.
7. What is the value of the average $\langle \sigma_i \sigma_j \sigma_k \rangle$ with $1 \leq i < j < k \leq N$? Generalize your answer.
8. Compute $\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle$, with $1 \leq i < j < k < l \leq N$, using the diagrammatic method. Simplify your result in the thermodynamic limit ($N \to \infty$ with all indices fixed).