ICFP M1 - Phase Transitions – TD n° 3
The Critical Temperature of the Ising Model in 2D

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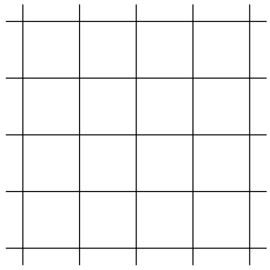
2023-2024

We shall study in this problem the Ising model on a two-dimensional square lattice of size $L \times L$, a portion of which is shown on the figure below :



The degrees of freedom of this model are $N = L^2$ Ising spins $\sigma_i = \pm 1$ that are placed on the vertices of this lattice, their global configurations being denoted $\underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$, with an energy given by

$$H(\underline{\sigma}) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \ , \tag{1}$$

where the sum runs over all links of the lattice (or equivalently all pairs of nearest neighbours), we assume periodic boundary conditions in both directions, and $J > 0$ (the interactions are ferromagnetic). The goal of the problem is to determine the critical temperature $T_{\rm c}$ of the model, by a duality argument due to Kramers and Wannier.

1 High temperature expansion

1. How many terms are there in the sum in equation (1) ?
2. Show that $e^{\beta J \sigma_i \sigma_j} = c(1 + t\sigma_i\sigma_j)$ and give explicit expressions for the constants c and t.
3. Using this identity, show that the partition function at inverse temperature β can be rewritten as :
$$Z_N(\beta) = c^{2N} \sum_{\underline{\sigma}} \prod_{\langle ij \rangle} (1 + t\sigma_i\sigma_j) \ . \tag{2}$$
How many terms are there in the expansion of the product ?
4. Let us associate each term in the expansion of the product with a diagram, that is a subset of those links on the lattice that correspond to the the factors $t\sigma_i\sigma_j$ appearing in the term. Which diagrams contribute to equation (2) after having summed over the configurations $\underline{\sigma}$ of the spins ?
5. We will order the expansion of equation (2) into powers of t according to
$$Z_N(\beta) = (2c^2)^N \sum_{n=0}^{\infty} a_{N,n} t^n \ , \tag{3}$$
Specify in words and without any calculation the value of the coefficients $a_{N,n}$ and justify the name "high temperature expansion" given to this series. In the following we will denote $A_N(x) = \sum_n a_{N,n} x^n$.
6. Compute the values $a_{N,n}$ for $n = 0, 1, \ldots, 6$.

2 Low temperature expansion

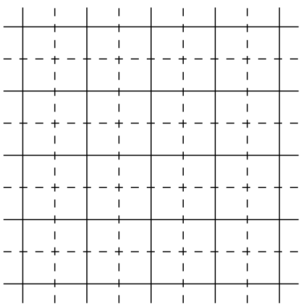
7. What is the order of magnitude that discriminates "high" from "low" temperatures?
8. Which configurations minimize the Hamiltonian (1)? Give their energy E_0 and their degeneracy.
9. What is the energy $E_1 > E_0$ of the first excited state? Describe the corresponding spin configurations and give their degeneracy.
10. Same question for the second excited state with energy $E_2 > E_1$.
11. From these considerations deduce the following low energy expansion of the partition function,

$$Z_N(\beta) = 2e^{2N\beta J}\left(b_{N,0} + b_{N,4}\left(e^{-2\beta J}\right)^4 + b_{N,6}\left(e^{-2\beta J}\right)^6 + o\left(\left(e^{-2\beta J}\right)^6\right)\right) , \quad (4)$$

and specify the values of the coefficients $b_{N,n}$. Compare these coefficients to those of the high temperature expansion, i.e. to $a_{N,n}$.
12. Show that the low energy expansion of the partition function can be rewritten as

$$Z_N(\beta) = 2e^{2N\beta J} A_N(e^{-2\beta J}),$$

where $A_N(x) = \sum_n a_{N,n} x^n$ as already defined in question 5. To do so, consider graphs on the dual lattice that is represented by dashed lines in the following figure.



3 Critical temperature

13. Let us denote by $f(\beta) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$ the free energy per spin in the thermodynamic limit. Deduce two expressions for $f(\beta)$, one from the high and one from the low temperature expansions of the partition function. You can express your result in terms of $g(x) = \lim_{N\to\infty} \frac{1}{N} \log A_N(x)$.
14. Assuming that the model has a unique critical temperature and therefore that $f(\beta)$ is singular at a unique point β_c, show that the inverse critical temperature is given by

$$\beta_c J = \frac{1}{2}\log(1+\sqrt{2}) . \quad (5)$$

4 Exact results

15. An exact computation, first performed by Onsager, yields the following expression for $f(\beta)$:

$$f(\beta) = -\frac{1}{2\beta}\int_{-\pi}^{\pi} \frac{\mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^2} \log\left[(\cosh(2\beta J))^2 - (\sinh(2\beta J))(\cos(k_x) + \cos(k_y))\right] . \quad (6)$$

Check that $f(\beta)$ is indeed singular at the inverse critical temperature β_c determined above.
16. Another computation, due to Yang, gives the spontaneous magnetization for $T < T_c$ as:

$$m_{\mathrm{sp}}(T) = \left(1 - (\sinh(2\beta J))^{-4}\right)^{\frac{1}{8}} . \quad (7)$$

Deduce the value of the critical exponent β.