ICFP M1 - Phase Transitions – TD n° 4
Correlation Inequalities

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We consider a system of N Ising spins, of configurations denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$, and of energy

$$H(\underline{\sigma}) = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} J_{i,j} \sigma_i \sigma_j - \sum_{i=1}^{N} h_i \sigma_i \ . \tag{1}$$

For a subset of the indices $X \subset \{1, \dots, N\}$ we define $\sigma_X = \prod_{i \in X} \sigma_i$ the product of the spins in this subset. If X is a singleton $\{i\}$ we can use both notations $\sigma_{\{i\}} = \sigma_i$.

We write $\langle \bullet \rangle$ the average on the set of configurations with a Boltzmann weight and an inverse temperature β, i.e. for a function $f(\underline{\sigma})$ we have

$$\langle f(\underline{\sigma}) \rangle = \sum_{\underline{\sigma}} f(\underline{\sigma}) \frac{e^{-\beta H(\underline{\sigma})}}{Z} \ , \qquad Z = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})} \ . \tag{2}$$

1. What is the physical meaning of the constants $J_{i,j}$ and h_i ? From now on we asume $J_{i,j} \geq 0 \ \forall \, i, j$; what does this mean?

2. Assume $h_i \geq 0 \ \forall i \in \{1, \dots, N\}$. Show that for all subsets $X \subset \{1, \dots, N\}$,

$$\langle \sigma_X \rangle \geq 0 \ . \tag{3}$$

   Hint : Expand the exponential in the numerator of $\langle \sigma_X \rangle$ as a power series.

3. Using the result of the previous question (assuming $h_i \geq 0 \ \forall i$), show that for all tuples of subsets X, Y ($X \subset \{1, \dots, N\}$, $Y \subset \{1, \dots, N\}$) we have the Griffiths inequality:

$$\langle \sigma_X \sigma_Y \rangle \geq \langle \sigma_X \rangle \langle \sigma_Y \rangle \ . \tag{4}$$

   Hint : One can notice that

$$\langle \sigma_X \sigma_Y \rangle - \langle \sigma_X \rangle \langle \sigma_Y \rangle = \frac{1}{Z^2} \sum_{\underline{\sigma}, \underline{\sigma}'} (\sigma_X \sigma_Y - \sigma_X \sigma'_Y) e^{-\beta (H(\underline{\sigma}) + H(\underline{\sigma}'))} \tag{5}$$

   and make a change of variable $\tau_i = \sigma_i \sigma'_i$.

4. Let us consider 2 choices $\{h_i^{(1)}, i \in \{1, \dots, N\}\}$ and $\{h_i^{(2)}, i \in \{1, \dots, N\}\}$ of the parameters h_i, and write $\langle \bullet \rangle_1$ and $\langle \bullet \rangle_2$ the averages with respect to these two choices ($J_{i,j}$ being the same in both cases). Using the Fluctuation-Dissipation Theorem and the inequality (4), show that

$$0 \leq h_i^{(1)} \leq h_i^{(2)} \ \ \forall i \in \{1, \dots, N\} \quad \Rightarrow \quad \langle \sigma_X \rangle_1 \leq \langle \sigma_X \rangle_2 \ , \tag{6}$$

   whatever the choice of $X \subset \{1, \dots, N\}$.

5. Take the limit of this result for some h_i going to $+\infty$; what is the physical meaning of this limit?

6. How does $\langle \sigma_X \rangle$ vary with respect to the coupling constants $J_{i,j}$ (assuming $h_i \geq 0$)?

7. From this, deduce how $\langle \sigma_X \rangle$ varies with β, for a given value of $J_{i,j} \geq 0$ and $h_i \geq 0$.

8. Application of question 6 : Peierl's argument seen in the lectures shows that in dimension $d = 2$, the spontaneous magnetization is strictly positive for low enough but finite temperatures. Show that this property also holds in $d \geq 3$. *Hint:* For the 3d Ising model consider the case where some coupling constants $J_{i,j}$ are sent to zero such that the model splits in independent parallel 2d Ising models.

9. Application of question 6 : compare the values of T_c for the three regular 2d lattices, namely the triangular, honeycomb and square lattices.