ICFP M1 - Phase Transitions – TD n° 5 – Solution
Real-Space Renormalization Group

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1. The height of an equilateral triangle with edges of length a is $a\frac{\sqrt{3}}{2}$, hence $\widehat{a} = a\sqrt{3}$ and $b = \sqrt{3}$. There are $\widehat{N} = N/3$ spins in the new lattice, as each block contains 3 spins of the original lattice, and each of the original spin is in exactly one block. The decimation rule is applied independently in each block, with the correspondance

$$\widehat{\sigma}_\alpha = +1 \Leftrightarrow (\sigma_{i(\alpha)}, \sigma_{j(\alpha)}, \sigma_{k(\alpha)}) \in \{(+1,+1,+1),(-1,+1,+1),(+1,-1,+1),(+1,+1,-1)\} \ , \tag{1}$$
$$\widehat{\sigma}_\alpha = -1 \Leftrightarrow (\sigma_{i(\alpha)}, \sigma_{j(\alpha)}, \sigma_{k(\alpha)}) \in \{(-1,-1,-1),(+1,-1,-1),(-1,+1,-1),(-1,-1,+1)\} \ . \tag{2}$$

For each possible value of the new spin $\widehat{\sigma}_\alpha$ there are thus 4 allowed configurations for the three original spins of the block, hence $|C(\underline{\widehat{\sigma}})| = 4^{\widehat{N}} = 4^{N/3}$.

2. The definition of $\widehat{H}$ has been chosen in order to impose the equality of the partition functions before and after the decimation. Indeed,

$$\widehat{Z} = \sum_{\underline{\widehat{\sigma}}} e^{-\beta \widehat{H}(\underline{\widehat{\sigma}})} = \sum_{\underline{\widehat{\sigma}}} \sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})} = Z \ .$$

The probability of a configuration $\underline{\widehat{\sigma}}$ in the new system is the sum of the probabilities of the configurations of the original system that are decimated into $\underline{\widehat{\sigma}}$, i.e. those of $C(\underline{\widehat{\sigma}})$:

$$P_{\widehat{H}}(\underline{\widehat{\sigma}}) = \frac{1}{\widehat{Z}} e^{-\beta \widehat{H}(\underline{\widehat{\sigma}})} = \frac{1}{Z} \sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma} \in C(\underline{\widehat{\sigma}})} P_H(\underline{\sigma}) \ .$$

3. (a) A function $f(\widehat{\sigma}_1)$ of a single Ising spin can be written as a polynomial of $\widehat{\sigma}_1$ of degree 1, as

$$f(\widehat{\sigma}_1) = f(+1)\frac{1+\widehat{\sigma}_1}{2} + f(-1)\frac{1-\widehat{\sigma}_1}{2} = A_\emptyset + A_1 \widehat{\sigma}_1$$
$$\text{with} \quad A_\emptyset = \frac{f(+1)+f(-1)}{2} \ , \quad A_1 = \frac{f(+1)-f(-1)}{2} \ .$$

Reasoning by induction one can generalize this representation for functions of n Ising spins as

$$f(\widehat{\sigma}_1, \dots, \widehat{\sigma}_n) = \sum_{S \subset \{1,\dots,n\}} A_S \prod_{i \in S} \widehat{\sigma}_i \ , \quad \text{with } A_S = \frac{1}{2^n} \sum_{\widehat{\sigma}_1,\dots,\widehat{\sigma}_n} f(\widehat{\sigma}_1,\dots,\widehat{\sigma}_n) \prod_{i \in S} \widehat{\sigma}_i \ .$$

There is a bijective correspondence between the 2^n values of $f(\widehat{\sigma}_1,\dots,\widehat{\sigma}_n)$ and the 2^n values of A_S when S runs over the subsets of $\{1,\dots,n\}$; to convince yourself of this equivalence you can use the identity

$$\frac{1}{2^n} \sum_{\widehat{\sigma}_1,\dots,\widehat{\sigma}_n} \left(\prod_{i \in S} \widehat{\sigma}_i \right) \left(\prod_{i \in S'} \widehat{\sigma}_i \right) = \delta_{S,S'}$$

for two subsets S and S' of indices. Another way of picturing this representation is to expand f as a generic power series for real variables and simplify it using $\widehat{\sigma}_i^{2n} = 1$ and $\widehat{\sigma}_i^{2n+1} = \widehat{\sigma}_i$ for Ising spins.
In the present case we can hence write

$$\widehat{H}(\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3) = A_\emptyset + A_1\widehat{\sigma}_1 + A_2\widehat{\sigma}_2 + A_3\widehat{\sigma}_3 + A_{12}\widehat{\sigma}_1\widehat{\sigma}_2 + A_{13}\widehat{\sigma}_1\widehat{\sigma}_3 + A_{23}\widehat{\sigma}_2\widehat{\sigma}_3 + A_{123}\widehat{\sigma}_1\widehat{\sigma}_2\widehat{\sigma}_3 \ .$$

(b) According to the inversion formula stated above we can obtain the coefficients A with

$$A_S = -\frac{1}{\beta}\frac{1}{8} \sum_{\widehat{\sigma}_1,\widehat{\sigma}_2,\widehat{\sigma}_3} \left(\prod_{i\in S} \widehat{\sigma}_i \right) \ln \left[\sum_{\underline{\sigma}\in C(\widehat{\underline{\sigma}})} e^{-\beta H(\underline{\sigma})} \right] ,$$

which could be implemented in a symbolic computation software.

(c) In general all the coefficients A_S will be non-zero, which means that the decimation has generated a pairwise interaction of range 2 ($A_{13} \neq 0$) and an interaction between triplets of spins ($A_{123} \neq 0$) that were not present in the original Hamiltonian. One way to show that these coefficients are non-zero is to compute them in perturbation for small J_1 and J_2, one finds (after a rather cumbersome computation) that A_{13} is of order J_1^2, and A_{123} of order $J_1^2 J_2$.

4. One has

$$e^{-\beta H(\widehat{\underline{\sigma}})} = \sum_{\underline{\sigma}\in C(\widehat{\underline{\sigma}})} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma}\in C(\widehat{\underline{\sigma}})} e^{-\beta(H(\underline{\sigma})-H_0(\underline{\sigma}))} e^{-\beta H_0(\underline{\sigma})} = Z_0(\widehat{\underline{\sigma}}) \langle e^{-\beta(H(\underline{\sigma})-H_0(\underline{\sigma}))} \rangle_{0,\widehat{\underline{\sigma}}} .$$

The inequality comes from Jensen inequality, that states that for a convex function f the average of the function is larger than the function of the average, i.e. $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$, for any random variable X with an expectation denoted $\mathbb{E}$. Applying it here with $f(x) = e^x$, that is indeed convex, and $\langle \bullet \rangle_{0,\widehat{\underline{\sigma}}}$ playing the role of $\mathbb{E}[\bullet]$, one obtains

$$e^{-\beta H(\widehat{\underline{\sigma}})} \geq Z_0(\widehat{\underline{\sigma}}) e^{-\beta \langle H(\underline{\sigma})-H_0(\underline{\sigma}) \rangle_{0,\widehat{\underline{\sigma}}}} .$$

Taking logarithm and dividing by β (which reverses the direction of the inequality) one obtains the equation of the text,

$$\widehat{H}(\widehat{\underline{\sigma}}) \leq -\frac{1}{\beta} \ln Z_0(\widehat{\underline{\sigma}}) + \langle H(\underline{\sigma}) - H_0(\underline{\sigma}) \rangle_{0,\widehat{\underline{\sigma}}} . \tag{3}$$

The inequality becomes an equality if one takes $H_0(\underline{\sigma}) = H(\underline{\sigma})$, but in this case one is back to the original problem that is in general impossible to solve. The point of the variational method is to choose some trial Hamiltonian H_0 simple enough for the right hand side of the last equation to be computable exactly.

5. The crucial simplifying feature of this variational Hamiltonian H_0 is the absence of interaction between spins belonging to distinct blocks, the latter thus decouple one from the other. This leads for the computation of $Z_0(\widehat{\underline{\sigma}})$:

$$\begin{aligned} Z_0(\widehat{\underline{\sigma}}) &= e^{NJ_0} \prod_{\alpha=1}^{\widehat{N}} \left(\sum_{\substack{\sigma_{i(\alpha)},\sigma_{j(\alpha)},\sigma_{k(\alpha)} \\ \text{sign}\,(\sigma_{i(\alpha)}+\sigma_{j(\alpha)}+\sigma_{k(\alpha)})=\widehat{\sigma}_\alpha}} e^{J_1(\sigma_{i(\alpha)}\sigma_{j(\alpha)}+\sigma_{i(\alpha)}\sigma_{k(\alpha)}+\sigma_{j(\alpha)}\sigma_{k(\alpha)})} \right) \\ &= e^{NJ_0} (e^{3J_1} + 3\,e^{-J_1})^{\widehat{N}} . \end{aligned}$$

Indeed, among the four configurations allowed for the three spins inside the α-th block, given explicitly in equation (1), one of them has the three spins equal to $\widehat{\sigma}_\alpha$, yielding the term e^{3J_1}, and the three others have two spins equal to $\widehat{\sigma}_\alpha$, the last one equal to $-\widehat{\sigma}_\alpha$, which gives the term $3\,e^{-J_1}$. Note that $Z_0(\widehat{\underline{\sigma}})$ is independent of $\widehat{\underline{\sigma}}$.

The computation of $\langle \sigma_i \rangle_{0,\widehat{\underline{\sigma}}}$ is similar, denoting $\alpha(i)$ the block to which i belongs one can factor out the other blocks and write

$$\begin{aligned} \langle \sigma_i \rangle_{0,\widehat{\underline{\sigma}}} &= \frac{1}{e^{3J_1} + 3\,e^{-J_1}} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3 \\ \text{sign}\,(\sigma_1+\sigma_2+\sigma_3)=\widehat{\sigma}_{\alpha(i)}}} \sigma_1\, e^{J_1(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)} = \widehat{\sigma}_{\alpha(i)} \frac{e^{3J_1} + (2-1)e^{-J_1}}{e^{3J_1} + 3\,e^{-J_1}} \\ &= \widehat{\sigma}_{\alpha(i)} \frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\,e^{-J_1}} , \end{aligned}$$

exploiting the description of the four relevant configurations explained above.

If i and j belong to different blocks the spins σ_i and σ_j are independent under the Gibbs-Boltzman law induced by the Hamiltonian H_0, hence in this case

$$\langle \sigma_i \sigma_j \rangle_{0,\underline{\widehat{\sigma}}} = \langle \sigma_i \rangle_{0,\underline{\widehat{\sigma}}} \langle \sigma_j \rangle_{0,\underline{\widehat{\sigma}}} \ .$$

6. We can use now the results of the previous question to compute the right hand side of equation (3). The first term is easily obtained from

$$-\frac{1}{\beta} \ln Z_0(\underline{\widehat{\sigma}}) = -\frac{3 J_0 \widehat{N}}{\beta} - \frac{1}{\beta} \widehat{N} \ln(e^{3J_1} + 3 e^{-J_1}) \ .$$

Moreover in $H(\underline{\sigma}) - H_0(\underline{\sigma})$ the interactions inside the blocks cancel out, yielding

$$\begin{aligned}
\langle H(\underline{\sigma}) - H_0(\underline{\sigma}) \rangle_{0,\underline{\widehat{\sigma}}} &= \left\langle -\frac{J_1}{\beta} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \frac{J_2}{\beta} \sum_{i=1}^{N} \sigma_i + \frac{J_1}{\beta} \sum_{\alpha=1}^{\widehat{N}} \sum_{\langle i,j \rangle \in \alpha} \sigma_i \sigma_j \right\rangle_{0,\underline{\widehat{\sigma}}} \\
&= \left\langle -\frac{J_1}{\beta} \sum_{\substack{\langle i,j \rangle \\ \alpha(i) \neq \alpha(j)}} \sigma_i \sigma_j - \frac{J_2}{\beta} \sum_{i=1}^{N} \sigma_i \right\rangle_{0,\underline{\widehat{\sigma}}} \\
&= -\frac{J_1}{\beta} \sum_{\substack{\langle i,j \rangle \\ \alpha(i) \neq \alpha(j)}} \langle \sigma_i \sigma_j \rangle_{0,\underline{\widehat{\sigma}}} - \frac{J_2}{\beta} \sum_{i=1}^{N} \langle \sigma_i \rangle_{0,\underline{\widehat{\sigma}}} \\
&= -\frac{2 J_1}{\beta} \sum_{\langle \alpha,\beta \rangle} \widehat{\sigma}_\alpha \widehat{\sigma}_\beta \left(\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3 e^{-J_1}} \right)^2 - \frac{3 J_2}{\beta} \sum_{\alpha=1}^{\widehat{N}} \widehat{\sigma}_\alpha \frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3 e^{-J_1}} \ ,
\end{aligned}$$

where the factors 2 and 3 follow from noting that there are two edges between sites of the adjacent blocks α and β, and 3 sites in each block α. Plugging these two results in the right hand side of equation (3) one obtains the result of the text.

7. The fixed points are the solution of the following system of equations:

$$\begin{cases} J_1 = 2\ J_1 \left(\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\ e^{-J_1}} \right)^2 \\ J_2 = 3\ J_2 \left(\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\ e^{-J_1}} \right) \end{cases} .$$

The first one is independent of J_2; it admits the solution $J_1 = 0$, in which case the second one simplifies into $J_2 = \frac{3}{2} J_2$, which only admits $J_2 = 0$ as a solution. Suppose now that $J_1 \neq 0$ is solution of the first equation; then one must have

$$\frac{e^{3J_1} + e^{-J_1}}{e^{3J_1} + 3\ e^{-J_1}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad e^{4J_1} \left(1 - \frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}} - 1 \quad \Rightarrow \quad J_1 = \frac{1}{4} \ln(1 + 2\sqrt{2}) = J_{1c} \ .$$

Plugging this value of J_1 in the second equation yields $J_2 = \frac{3}{\sqrt{2}} J_2$, which only admits $J_2 = 0$ as a solution.

To summarize, there are 2 fixed points, a trivial one $(J_1^*, J_2^*) = (0,0)$ in which there is no interaction before and after the renormalization transformation, and a non-trivial one $(J_1^*, J_2^*) = (\frac{1}{4} \ln(1 + 2\sqrt{2}), 0)$, with no magnetic field but a non-zero interaction between neighboring spins.

To determine the stability of these fixed points one has to compute the eigenvalues of the Jacobian of the transformation, i.e. of the matrix

$$M = \begin{pmatrix} \frac{\partial \widehat{J}_1}{\partial J_1} & \frac{\partial \widehat{J}_2}{\partial J_1} \\ \frac{\partial \widehat{J}_1}{\partial J_2} & \frac{\partial \widehat{J}_2}{\partial J_2} \end{pmatrix} .$$

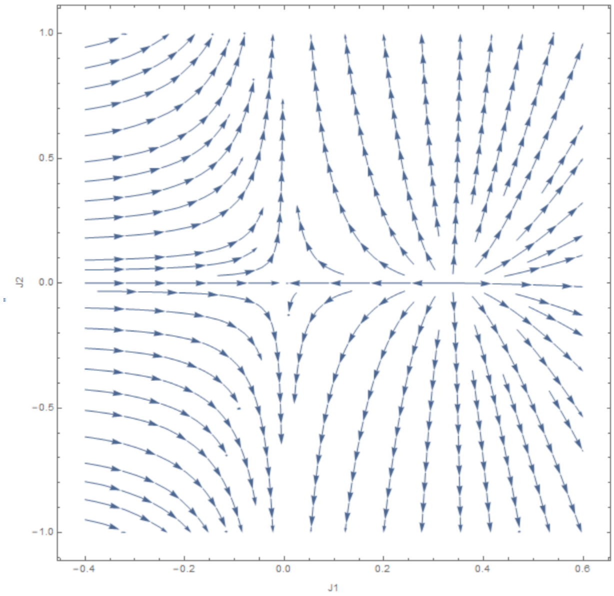

Figure 1: Flow of the renormalisation group transformations.

After a short computation one finds for this matrix, at the trivial and non-trivial fixed points respectively:

$$M_{\rm t}(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix} , \qquad M_{\rm nt} = \begin{pmatrix} 1 + \frac{8-5\sqrt{2}}{2} \ln(1+2\sqrt{2}) & 0 \\ 0 & 3/\sqrt{2} \end{pmatrix} .$$

These matrices being diagonal, their eigenvalues can be simply read as their diagonal elements. At each step of the renormalization transformation, a vector of parameters (J_1, J_2) close to a given fixed point is multiplied by the associated Jacobian matrix M. After n steps it gets multiplied by M^n, hence an eigenvalue larger than 1 leads to a growth in the corresponding eigenvector direction, which is thus unstable. On the contrary an eigenvalue smaller than 1 corresponds to a stable direction, leading to a contraction under the iterations. The trivial fixed point is stable in the J_1 direction and unstable in the J_2 one: indeed a magnetic field breaks explicitly the spin-flip symmetry and gets amplified by the renormalization transformations. On the other hand the non-trivial fixed point is unstable in both directions, see Fig. 1 for an illustration. Consider in particular the behavior along the $J_2 = 0$ axis: values of $J_1 \in (0, J_{1c})$ are attracted towards $J_1 = 0$ by the iterations, this high temperature phase leads to a macroscopic non-interacting effective model. On the contrary the low temperature phase $J_1 > J_{1c}$ sees its coupling constant grow under iterations, the non-trivial fixed point is thus a critical point separating these two behaviors.

8. Writing $\lambda_1 = 1 + \frac{8-5\sqrt{2}}{2} \ln(1+2\sqrt{2}) = b^{y_1}$ and $\lambda_2 = \frac{3}{\sqrt{2}} = b^{y_2}$ with the scale factor $b = \sqrt{3}$ one finds the numerical values $y_1 \approx 0,882$ and $y_2 \approx 1,368$. The critical exponents are then deduced from the formulas of the text,

$$\alpha \approx -0,267 \ , \quad \beta \approx 0,716 \ , \quad \gamma \approx 0,834 \ , \quad \delta \approx 2,165 \ , \quad \nu \approx 1,134 \ , \quad \eta \approx 1,264 \ .$$

The exact value are $y_1 = 1$ and $y_2 = 15/8 = 1,875$; in the approximation performed in this problem it is y_2 that is the most inexact, hence the exponents that only depend on y_1 are relatively better evaluated.