ICFP M1 - Phase Transitions – TD n° 6
Scaling Functions and Relationships between Critical Exponents

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Let us consider a ferromagnetic system at temperature T in a magnetic field h, and denote m its magnetization per spin. We shall assume that it undergoes a phase transition at the critical temperature T_c, and parametrize the distance to the critical temperature as $\beta = \beta_\mathrm{c}(1+\epsilon)$ ($\epsilon > 0$ thus corresponds to the low temperature phase). Recall the definition of the critical exponents β, δ, γ ($\propto$ denotes an equivalent modulo a multiplicative constant):

- the spontaneous magnetization behaves as $m(\epsilon, h = 0^+) \propto \epsilon^\beta$ when $\epsilon \to 0^+$.
- right at the critical temperature, $m(\epsilon = 0, h) \propto h^{\frac{1}{\delta}}$ when $h \to 0^+$.
- the zero-field susceptibility behaves as $\chi = \frac{\partial m}{\partial h}\Big|_{h=0} \propto \begin{cases} \epsilon^{-\gamma'} & \text{when } \epsilon \to 0^+ \\ (-\epsilon)^{-\gamma} & \text{when } \epsilon \to 0^- \end{cases}$

1. Check that these definitions agree with the usual ones in terms of the reduced temperature $t = \frac{T - T_\mathrm{c}}{T_\mathrm{c}}$.
2. Draw the shape of the curves of the magnetization as a function of h, for $\epsilon = 0, < 0$ et > 0. Draw on another graph the shape of the magnetization as a function of ϵ, for $h = 0^+$ and $h > 0$.

1 The mean-field value of the critical exponents

In the mean-field description the magnetization $m(\epsilon, h)$ is the (greatest when $h \geq 0$) solution of the implicit equation

$$m = \tanh((1+\epsilon)(m+h)) \ . \tag{1}$$

We use here adimensional quantities, in such a way that $\beta_\mathrm{c} = 1$.

3. We consider first $h = 0^+$. Find the value of the exponent β, by expanding the equation (1) at the lowest non-trivial order.
4. Do the same at $\epsilon = 0$ to determine the exponent δ.
5. Express the susceptibility in terms of $m(\epsilon, h)$; deduce from it the value of the exponents γ and γ'.

2 Existence of a scaling function in mean-field

We shall now study the matching between the regime $\epsilon = 0$, $h \to 0^+$ governed by the exponent δ and the regime $h = 0^+$, $\epsilon \to 0^+$ governed by β. We shall thus suppose that ϵ and h tend simultaneously to 0.

6. What must be the relative orders of m and ϵ with respect to h for the terms of the expansions of the equation (1) kept in the questions 3 and 4 to be of the same order ?

7. Deduce from the previous question the value of the exponents a and b such that the function

$$g(u) = \lim_{h\to 0^+} \frac{m(\epsilon = u\, h^b, h)}{h^a} \tag{2}$$

is non-trivial. Such a function g is called a scaling function, and one denotes

$$m \sim h^a\, g\left(\frac{\epsilon}{h^b}\right) \tag{3}$$

in the regime where ϵ and h both go to 0 with ϵ/h^b finite.

8. Write the equation verified by $g(u)$ in mean-field. Draw the shape of this function, and specify its behaviour around $u = 0$ and $u = \pm\infty$.

3 General consequences of the existence of a scaling function

We shall assume now that the system is finite dimensional (with $d \geq 2$), and that there exists a scaling regime such that the relations (2,3) are valid (with a, b and g a priori distinct from the values they take in mean-field).

9. We consider first $\epsilon = 0$. What must be the behavior of g in $u = 0$? Find a relationship between a and δ.

10. Suppose that $g(u) \propto u^c$ when $u \to +\infty$; consider the limit $h \to 0^+$ with $\epsilon > 0$ fixed, and express the exponents b and c in terms of β and δ.

11. Follow a similar reasoning to determine the behavior of g in $-\infty$. Deduce the following relationship between the critical exponents: $\gamma = \beta(\delta - 1)$.

12. Write the susceptibility under the scaling form $\chi = \frac{\partial m}{\partial h} \sim h^{a'} \widehat{g}\left(\frac{\epsilon}{h^b}\right)$, specifying the value of a' and giving the expression of $\widehat{g}(u)$ in terms of $g(u)$.

13. Find the behavior of $\widehat{g}(u)$ in $+\infty$, and deduce the relationship $\gamma' = \beta(\delta - 1)$.

14. Check the consistency of these general results with those obtained from the mean-field equation.

4 Another relationship between critical exponents

The correlation function of a magnetic system is denoted $G_{i,j} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$, where i and j are two sites of the lattice on which the system is defined. Close to the critical point, and for distances r between the two sites much larger than the distance between neighboring sites (taken to be 1 for simplicity), this function admits a scaling form

$$G(r) \sim \frac{1}{r^{d-2+\eta}}\, g(r/\xi) \ , \tag{4}$$

where ξ is the correlation length of the system, $g(u)$ having an exponential decay for $u \gg 1$. The correlation length diverges at the transition with the exponent ν, i.e. $\xi \propto |\epsilon|^{-\nu}$.

15. Let us denote $M(V) = \sum_{i\in V} \sigma_i$ the total magnetization in a domain V of the lattice, containing $|V| = L^d$ sites. Close to the transition, in the low-temperature phase with positive magnetization, one denotes $m_{\text{sp}} \propto \epsilon^\beta$ the spontaneous magnetization per spin. What is the average value of $M(V)$?

16. Express the variance of $M(V)$ in terms of the correlation function $G_{i,j}$. Estimate its order of magnitude with respect to L for $1 \ll L \lesssim \xi$.

17. We assume that when L is of the order of the correlation length ξ, the average value and the root mean square of $M(V)$ are of the same order. Deduce from this assumption the following relationship between critical exponents, $\beta = \frac{\nu}{2}(d - 2 + \eta)$.

18. In which dimension d the mean-field exponents do satisfy this relationship ?