ICFP M1 - QUANTUM MATTER - TD n°5 - Solutions
Spin liquids & Chern numbers

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1 Quantum spin ice: emergent quantum electrodynamics

The pyrochlore lattice is a lattice of corner-sharing tetrahedra (see Fig. 1B). The underlying Bravais lattice is a Face Centered Cubic (FCC) lattice (see Fig. 1A), and the lattice basis is a regular tetrahedron. The unit cell is made of two tetrahedrons.

1. How many sites per unit cell are there?

There are four since a tetrahedron has four sites. We will label them 0, 1, 2, 3.

The diamond lattice is a lattice with a two-atom basis (see Fig. 1C). The underlying Bravais lattice is an FCC lattice, and, if a diamond lattice site sits at an FCC site, the second atom (other sublattice) sits a quarter of the way along the diagonal of the FCC cubic cell.

2. What lattice do the centers of the (nearest-neighbor) *bonds* of the diamond lattice form?

A pyrochlore lattice.

3. Where do the *sites* of the diamond lattice sit with respect to the *tetrahedra* of the pyrochlore lattice?

At the centers of the tetrahedra. The diamond lattice is said to be dual to the pyrochlore lattice.

Notice that

- there are two types of tetrahedra, the "up-pointing" tetrahedra and the "down-pointing" ones. This is consistent with the diamond lattice, which has *two* sites in its basis, being the dual of the pyrochlore lattice.
- the lattice decomposes into alternating kagomé (corner-sharing triangles in 2d) and triangular planes in various directions, and most importantly that the kagomé planes contain hexagons (loops of length 6), see Fig. 2.

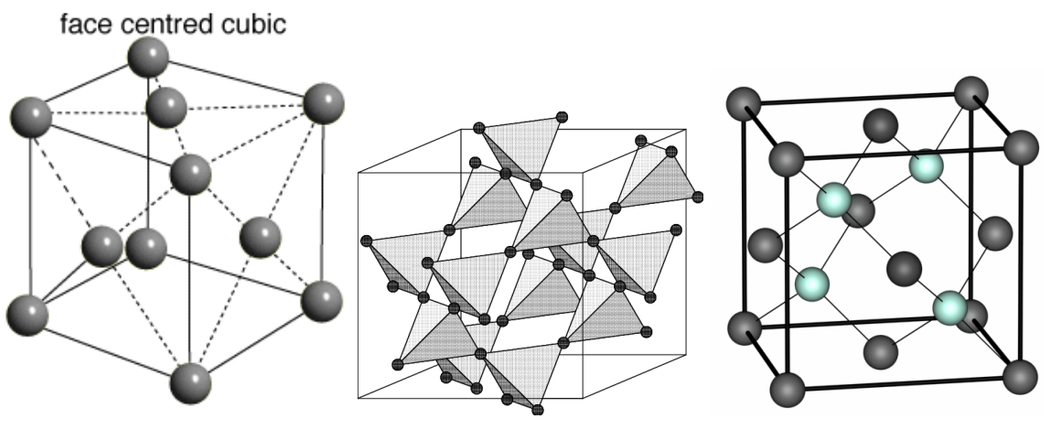

Figure 1: Some crystallographic structures. A: the face centerred cubic lattice. B: The pyrochlore lattice. C: The diamond lattice.

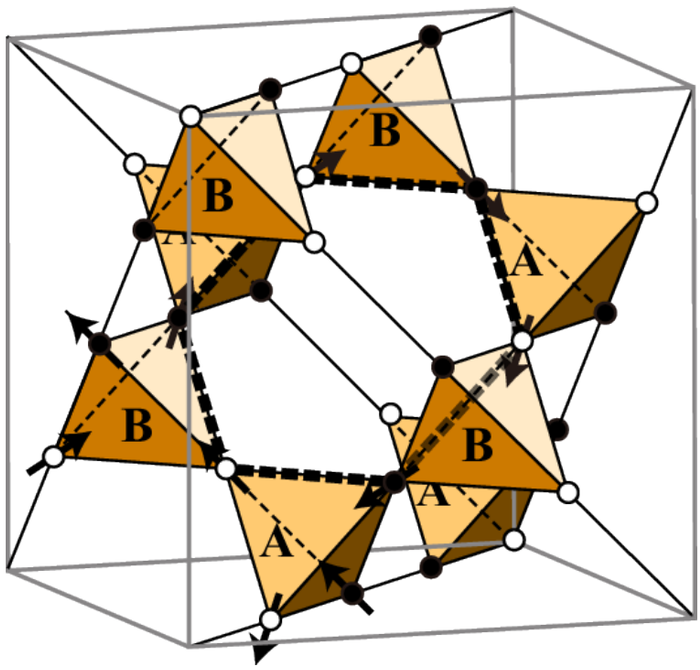

Figure 2: Hexagone loops in the pyrochlore lattice.

4. What is known as *classical* spin ice is the antiferromagnetic ($J_z > 0$) Ising model on the pyrochlore lattice. Rewrite the Ising terms on the bonds of a single tetrahedron as the square of a sum (up to a constant).

$$H = J_z \sum_{\langle ij \rangle} S_i^z S_j^z = \frac{J_z}{2} \sum_t (\sum_{i \in t} S_i^z)^2 + \text{const.} \tag{1}$$

where t is summed over the different tetrahedrons.

5. What is the condition which should be fulfilled by the ground state(s) (recall $J_z > 0$)?

We should have $\sum_{i \in t} S_i^z = 0$ for all tetrahedra, i.e.

$$S_0^z + S_1^z + S_2^z + S_3^z = 0. \tag{2}$$

6. For $S = 1/2$ find all (degenerate) ground states of a single tetrahedron.

Since $S_i^z = \pm 1/2$, this means that we want two spins up and two spins down on each tetrahedron. There are $C_4^2 = 6$ such possibilities.

7. If the tetrahedra were truly independent, what would the ground state degeneracy of a lattice of N unit cells be?

6^{2N} because there are two tetrahedra per unit cell.

We now add quantum fluctuations, *i.e.* terms in the Hamiltonian which do not commute with the Ising part. In particular, we consider "XY" or "transverse" terms:

$$H_\pm = -J_\pm \sum_{\langle ij \rangle} (S_i^+ S_j^- \pm S_i^- S_j^+). \tag{3}$$

The effective Hamiltonian in perturbation theory in small $J_\pm / J_z$ actually vanishes at first and second orders. The Hamiltonian at third order takes the form

$$H_{\text{eff}} = -K \sum_{\{i,j,k,l,m,n\}=\text{hexagon}} \left(S_i^+ S_j^- S_k^+ S_l^- S_m^+ S_n^- + \text{h.c.} \right) \tag{4}$$

8. How does K scale as a function of $J_\pm$ and J_z?

$K \sim J_\pm^3/J_z^2$ since this term is obtained at third order and K has the dimensions of J. In fact, one can show that $K = 12J_\pm^3/J_z^2$.

9. Show that indeed this Hamiltonian acts purely in the ground state manifold.

Applying $H_\pm$ once, one creates two defect tetrahedra around a hexagon. By applying it three times around the loop, the defect tetrahedra eventually "annihilate".

We will now show the relation of this to a $U(1)$ lattice gauge theory. Since the effective Hamiltonian operates within the space with $\sum_{i\in t} S_i^z = 0$, it necessarily commutes with $\sum_{i\in t} S_i^z$ for every tetrahedron t separately.

10. Where does this come from within H_{eff}?

One can carry out the calculation explicitly, but we may also notice that there is always one raising and one lowering spin operator in each tetrahedron in H_{eff}.

More formally, we may define

$$Q_t = \epsilon_t \sum_{i\in t} S_i^z \underset{0}{=} 0, \tag{5}$$

where for convenience we included the factor ϵ_t which assigns a sign that discriminates two orientations of tetrahedra: $\epsilon_t = +1(-1)$ for up (down) oriented tetrahedra. The equality on the right hand side of Eq. (5) holds in the ground state sector (as indicated by the subscript 0 on the equality sign). The "charge" Q_t commutes with H_{eff}. It follows that H_{eff} is invariant under the $U(1)$ symmetry generated by the unitary operator $U = e^{i\chi Q_t}$, for any t and any χ.

Noting that the charge Q_t is defined on the sites t of a diamond lattice (the tetrahedron centers), we can recast the effective Hamiltonian entirely as a diamond lattice gauge theory. To do so, we define

$$\begin{cases} S_{t,t'}^\pm = S_{i(t,t')}^{\pm\epsilon_t} \\ S_{t,t'}^z = S_{i(t,t')}^z \end{cases}, \tag{6}$$

where $i(t,t')$ is the pyrochlore site shared by neighboring tetrahedra t and t'. This definition gives a "vectorial" sense to the operators on the diamond lattice. Then the gauge charge becomes

$$Q_t = \epsilon_t \sum_{t'\in\partial t} S_{t,t'}^z \equiv \epsilon_t (\text{div}[S^z])_t \underset{0}{=} 0, \tag{7}$$

where the sum indicates nearest neighbor sites of t, and the result is the lattice divergence at t of the vector field $S_{t,t'}^z$. We recognize therefore $S_{t,t'}^z$ as the lattice analog of the "electric field" in Gauss' law. Let us make things even more explicit by writing things as:

$$S_{t,t'}^z \to E_{t,t'}, \qquad S_{t,t'}^\pm \to e^{\pm iA_{t,t'}}, \tag{8}$$

where $E_{t,t'}$ is a half-integral valued angular momentum, and $A_{t,t'} = -A_{t',t}$ is a 2π-periodic angular variable. If we impose the canonical commutation relations $[E_{t,t'}, A_{t,t'}] = i$ (think of $[L^z, \varphi] = i$), then $S_{t,t'}^\pm$ acts as the desired spin raising/lowering operator, except that it can raise/lower "outside" the physical space with $E_{t,t'} = \pm 1/2$.

11. Rewrite H_{eff} using Eq. (8). Defining the lattice curl as

$$(\text{curl}[A])_{\text{hexagon}} = \sum_{(t\in\text{hex})=1}^{6} A_{t,t+1} \tag{9}$$

which defines a "magnetic flux" B through a plaquette, show that H_{eff} takes the form of

$$H_{\text{eff}} = -2K \sum_{\text{hexagon}} \cos(\text{curl}[A]). \tag{10}$$

$$\begin{aligned} H_{\text{eff}} &= -K \sum_{\text{hexagon}} (e^{i(A_{1,2}-(-A_{2,3})+A_{3,4}-(-A_{4,5})+A_{5,6}-(-A_{6,1}))} + \text{h.c.}) && (11)\\ &= -2K \sum_{\text{hexagon}} \cos(A_{1,2}+A_{2,3}+A_{3,4}+A_{4,5}+A_{5,6}+A_{6,1}) && (12)\\ &= -2K \sum_{\text{hexagon}} \cos(\text{curl}[A]) && (13) \end{aligned}$$

12. Show that this Hamiltonian has a manifest gauge invariance under local phase rotations by an angle χ_t on each diamond site, for which

$$S^{\pm}_{t,t'} \to e^{\pm i(\chi_{t'}-\chi_t)} S^{\pm}_{t,t'}. \tag{14}$$

It is clear from the previous question: such a transform preserves the curl.

13. Now imposing the spin-1/2 constraint y adding a term $U(E^2 - 1/4)$ (notice that $U \to +\infty$ will project out the unphysical states), and assuming small $B = \mathrm{curl}[A]$, show that the Hamiltonian is that of quantum electrodynamics without matter fields $H \sim E^2 + B^2$.

$$H = \frac{\tilde{U}}{2} \sum_{\langle t,t' \rangle} E^2_{t,t'} + \frac{\tilde{K}}{2} \sum_{\mathrm{hex}} B^2, \tag{15}$$

where $\tilde{U} = 2U$, $\tilde{K} = 2 \times (-2K(-1/2)) = 2K$, since we assumed we could take $\cos(\mathrm{curl}A) \approx 1 - (\mathrm{curl}A)^2/2$. This is the Hamiltonian for quantum electrodynamics. It is a $U(1)$ gauge theory whose solutions we know. At low energies, there is a photon mode. Here the photon mode corresponds to fluctuations of S^z, and can indeed be seen in neutron scattering experiments through the $\langle S^z S^z \rangle$ correlations.

Remark: This describes the low energy theory, without matter fields (without "electrons" and "positrons"). To go further and include those, one instead writes $S^+_{r,r'} = \Phi^\dagger_t e^{iA_{t,t'}} \Phi_{t'}$ and S^z is unchanged, one obtains spinons (Φ) hopping in the background of fluctuating gauge fields, and the gauge transformation must include also $\Phi_t \to \Phi_t e^{-i\chi_t}$.

2 Berry curvature and Hall conductivity of Haldane's honeycomb model

In class, you introduced the low-energy Hamiltonian version of the Haldane model, namely

$$H = v\left(\mu^z \tau^x k_x + \tau^y k_y\right) + m_1 \tau^z + m_2 \mu^z \tau^z, \tag{16}$$

where the Pauli matrices mean the following:

- $\tau^z = \pm 1$: sublattices A/B,
- $\mu^z = \pm 1$: valley K/K',
- $\sigma^z = \pm 1$: spin $\pm 1/2$ (H in Eq. (16) is diagonal in spin, i.e. one should think about it as being multiplied by $\sigma^0 = \mathrm{Id}_2^{(\sigma)}$).

Using the fact that μ^z commutes with all terms in H, you found that the energy dispersions for fixed $\mu^z = \pm 1$ were

$$\epsilon_\pm(m_1, m_2) = \pm\sqrt{v^2k^2 + (m_1 + \mu^z m_2)^2}. \tag{17}$$

1. Derive this and find expressions for the corresponding eigenstates.

Let use write

$$\begin{aligned} H &= v\mu^z\tau^x k_x + v\tau^y k_y + m\tau^z && (18)\\ &= \mathbf{n}\cdot\boldsymbol{\tau}, && (19)\end{aligned}$$

where $m = m_1 + m_2\mu^z$ and

$$\mathbf{n} = (v\mu^z k_x, vk_y, m), \qquad |\mathbf{n}| = \sqrt{v^2k^2 + m^2}, \tag{20}$$

and whose eigenvalues and eigenvectors can be taken to be $\epsilon_\pm = \pm|\mathbf{n}|$,

$$\mathbf{u}_+ = \frac{1}{\sqrt{2|\mathbf{n}|(|\mathbf{n}| + n_z)}} \begin{pmatrix} n_z + |\mathbf{n}| \\ n_x + in_y \end{pmatrix}, \qquad \mathbf{u}_- = \frac{1}{\sqrt{2|\mathbf{n}|(|\mathbf{n}| + n_z)}} \begin{pmatrix} -n_x + in_y \\ n_z + |\mathbf{n}| \end{pmatrix} \tag{21}$$

2. Show that the gap vanishes (i.e. there exists at least one $\mathbf{k}_0$ such that $\epsilon_-(\mathbf{k}_0) = \epsilon_+(\mathbf{k}_0)$) if $|m_1| = |m_2|$ and in the case $\mu^z = -\mathrm{sign}(m_1/m_2)$.

We just plug that in and indeed find that the two bands touch at $k = 0$.

Assume that the system is "half-filled," i.e. that the ϵ_- band(s) are filled, and the ϵ_+ one(s) are empty. Recall the expression of the Berry phase of the lower band (ϵ_- band, also called valence band):

$$\Omega_-(\mathbf{k}) = \partial_x \mathcal{A}_-^y - \partial_y \mathcal{A}_-^x, \qquad \text{with} \qquad \mathcal{A}_-^\mu(\mathbf{k}) = i\langle u_{-\mathbf{k}}|\partial_\mu|u_{-\mathbf{k}}\rangle, \tag{22}$$

where $\partial_\mu \equiv \frac{\partial}{\partial k^\mu}$. We will derive an expression of the Berry curvature (which applies beyond the Haldane model) which will be easier to use than the definition Eq. (22).

3. Show that, for $n = \pm$,

$$\Omega_n(\mathbf{k}) = -2\mathrm{Im}[\langle \partial_x u_{n\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle]. \tag{23}$$

$$\begin{aligned}
\Omega_n(\mathbf{k}) &= i(\partial_x\langle u_n(\mathbf{k})|\partial_y u_n(\mathbf{k})\rangle - \partial_y\langle u_n(\mathbf{k})|\partial_x u_n(\mathbf{k})\rangle) && (24)\\
&= i(\langle \partial_x u_n(\mathbf{k})|\partial_y u_n(\mathbf{k})\rangle + \langle u_n(\mathbf{k})|\partial_x\partial_y u_n(\mathbf{k})\rangle - \langle \partial_y u_n(\mathbf{k})|\partial_x u_n(\mathbf{k})\rangle - \langle u_n(\mathbf{k})|\partial_y\partial_x u_n(\mathbf{k})\rangle) && (25)\\
&= i(\langle \partial_x u_n(\mathbf{k})|\partial_y u_n(\mathbf{k})\rangle - \langle \partial_y u_n(\mathbf{k})|\partial_x u_n(\mathbf{k})\rangle) && (26)\\
&= -2\mathrm{Im}[\langle \partial_x u_{n\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle] && (27)
\end{aligned}$$

4. Using $\langle u_{n\mathbf{k}}|u_{n\mathbf{k}}\rangle = 1$ and $\langle u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle = 0$ for $m \neq n$ and differentiating, show that

$$\Omega_n(\mathbf{k}) = -i\sum_{m\neq n}(\langle u_{n\mathbf{k}}|\partial_x u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle - \langle u_{n\mathbf{k}}|\partial_y u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x u_{n\mathbf{k}}\rangle) \tag{28}$$

$$\begin{aligned}
\Omega_n(\mathbf{k}) &= i(\langle \partial_x u_n(\mathbf{k})|\partial_y u_n(\mathbf{k})\rangle - \langle \partial_y u_n(\mathbf{k})|\partial_x u_n(\mathbf{k})\rangle) && (29)\\
&= i\sum_m(\langle \partial_x u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle - \langle \partial_y u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x u_{n\mathbf{k}}\rangle) && (30)\\
&= i\sum_m(-\langle u_{n\mathbf{k}}|\partial_x u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle + \langle u_{n\mathbf{k}}|\partial_y u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x u_{n\mathbf{k}}\rangle) && (31)\\
&= -i\sum_{m\neq n}(\langle u_{n\mathbf{k}}|\partial_x u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y u_{n\mathbf{k}}\rangle - \langle u_{n\mathbf{k}}|\partial_y u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x u_{n\mathbf{k}}\rangle) && (32)
\end{aligned}$$

where we used

$$\partial_\mu(\langle u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle) = 0 \Longrightarrow \langle \partial_\mu u_{n\mathbf{k}}|u_{m\mathbf{k}}\rangle = -\langle u_{n\mathbf{k}}|\partial_\mu u_{m\mathbf{k}}\rangle \tag{33}$$

as well as the cancellation of the "diagonal" terms.

5. Show that

$$\Omega_n(\mathbf{k}) = i\sum_{m\neq n}\left(\frac{\langle u_{n\mathbf{k}}|\partial_x H|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y H|u_{n\mathbf{k}}\rangle - \langle u_{n\mathbf{k}}|\partial_y H|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x H|u_{n\mathbf{k}}\rangle}{(E_n(\mathbf{k}) - E_m(\mathbf{k}))^2}\right). \tag{34}$$

To do so, write Schrödinger's equation for a band n

$$H|u_{n\mathbf{k}}\rangle = E_n(\mathbf{k})|u_{n\mathbf{k}}\rangle, \tag{35}$$

contract it on the left with the eigenstate for a band m

$$\langle u_{m\mathbf{k}}|H|u_{n\mathbf{k}}\rangle = E_n(\mathbf{k})\langle u_{m\mathbf{k}}|u_{n\mathbf{k}}\rangle, \tag{36}$$

and differentiate with respect to k_μ.

For $m \neq n$,

$$\begin{aligned}
\partial_\mu(\langle u_{m\mathbf{k}}|H|u_{n\mathbf{k}}\rangle) &= 0 && (37)\\
\langle \partial_\mu u_{m\mathbf{k}}|H|u_{n\mathbf{k}}\rangle + \langle u_{m\mathbf{k}}|\partial_\mu H|u_{n\mathbf{k}}\rangle + \langle u_{m\mathbf{k}}|H|\partial_\mu u_{n\mathbf{k}}\rangle &= 0 && (38)\\
E_n(\mathbf{k})\langle \partial_\mu u_{m\mathbf{k}}|u_{n\mathbf{k}}\rangle + \langle u_{m\mathbf{k}}|\partial_\mu H|u_{n\mathbf{k}}\rangle + E_m(\mathbf{k})\langle u_{m\mathbf{k}}|\partial_\mu u_{n\mathbf{k}}\rangle &= 0 && (39)\\
\langle u_{m\mathbf{k}}|\partial_\mu H|u_{n\mathbf{k}}\rangle &= (E_n(\mathbf{k}) - E_m(\mathbf{k}))\langle u_{m\mathbf{k}}|\partial_\mu u_{n\mathbf{k}}\rangle && (40)
\end{aligned}$$

So, for $E_m \neq E_n$,

$$\langle u_{m\mathbf{k}}|\partial_\mu u_{n\mathbf{k}}\rangle = \frac{\langle u_{m\mathbf{k}}|\partial_\mu H|u_{n\mathbf{k}}\rangle}{E_n(\mathbf{k}) - E_m(\mathbf{k})}, \tag{41}$$

and substituting twice, we obtain

$$\begin{aligned}\Omega_n(\mathbf{k}) &= -i\sum_{m\neq n}\left(\frac{\langle u_{n\mathbf{k}}|\partial_x H|u_{m\mathbf{k}}\rangle}{E_m(\mathbf{k}) - E_n(\mathbf{k})}\frac{\langle u_{m\mathbf{k}}|\partial_y H|u_{n\mathbf{k}}\rangle}{E_n(\mathbf{k}) - E_m(\mathbf{k})} - \frac{\langle u_{n\mathbf{k}}|\partial_y H|u_{m\mathbf{k}}\rangle}{E_m(\mathbf{k}) - E_n(\mathbf{k})}\frac{\langle u_{m\mathbf{k}}|\partial_x H|u_{n\mathbf{k}}\rangle}{E_n(\mathbf{k}) - E_m(\mathbf{k})}\right) &(42)\\ &= i\sum_{m\neq n}\left(\frac{\langle u_{n\mathbf{k}}|\partial_x H|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_y H|u_{n\mathbf{k}}\rangle - \langle u_{n\mathbf{k}}|\partial_y H|u_{m\mathbf{k}}\rangle\langle u_{m\mathbf{k}}|\partial_x H|u_{n\mathbf{k}}\rangle}{(E_n(\mathbf{k}) - E_m(\mathbf{k}))^2}\right). &(43)\end{aligned}$$

6. Let us now go back to the specific case of the Haldane model. Writing $H = v\mu^z\tau^x k_x + v\tau^y k_y + m\tau^z$ (think $m = m_1 + \mu^z m_2$) and noticing that

$$|u_{\pm\mathbf{k}}\rangle\langle u_{\pm\mathbf{k}}| = \frac{1}{2}\left(\tau^0 \pm \frac{H}{\epsilon_{+\mathbf{k}}}\right), \tag{44}$$

recover the result for the Berry phase of the lower band of the Haldane model given in class, i.e.

$$\Omega_-(\mathbf{k}) = \mu^z\frac{mv^2}{2(v^2k^2 + m^2)^{3/2}}. \tag{45}$$

We first compute $\partial_\mu H$:

$$\begin{aligned}\partial_x H &= v\mu^z\tau^x &(46)\\ \partial_y H &= v\tau^y &(47)\end{aligned}$$

The Berry phase is now:

$$\begin{aligned}\Omega_- &= \frac{i}{4\epsilon_+^2}\left[\langle u_{-\mathbf{k}}|\partial_x H|u_{+\mathbf{k}}\rangle\langle u_{+\mathbf{k}}|\partial_y H|u_{-\mathbf{k}}\rangle - \langle u_{-\mathbf{k}}|\partial_y H|u_{+\mathbf{k}}\rangle\langle u_{+\mathbf{k}}|\partial_x H|u_{-\mathbf{k}}\rangle\right] &(48)\\ &= \frac{i}{16\epsilon_+^2}\mathrm{Tr}\left[\partial_x H\left(\tau^0 + \frac{H}{\epsilon_{+\mathbf{k}}}\right)\partial_y H\left(\tau^0 - \frac{H}{\epsilon_{+\mathbf{k}}}\right) - \partial_x H\left(\tau^0 - \frac{H}{\epsilon_{+\mathbf{k}}}\right)\partial_y H\left(\tau^0 + \frac{H}{\epsilon_{+\mathbf{k}}}\right)\right] &(49)\\ &= \frac{i}{8\epsilon_+^3}\mathrm{Tr}\left[\partial_x HH\partial_y H - \partial_x H\partial_y HH\right] &(50)\\ &= \frac{iv^2\mu^z m}{8\epsilon_+^3}\mathrm{Tr}\left[\tau^x\tau^z\tau^y - \tau^x\tau^y\tau^z\right] &(51)\\ &= -\frac{iv^2\mu^z m}{4\epsilon_+^3}\mathrm{Tr}\left[\tau^x\tau^y\tau^z\right] &(52)\\ &= \frac{v^2\mu^z m}{2\epsilon_+^3} &(53)\end{aligned}$$

where we have used that $\tau_\alpha^2 = \tau^0$ and $\mathrm{Tr}[\tau_\alpha] = 0$ for $\alpha \in \{x, y, z\}$, and $\tau^x\tau^y\tau^z = i\tau^0$.

7. What is the sign of $\Omega_+(\mathbf{k})$ compared to that of $\Omega_-(\mathbf{k})$?

It's opposite.

8. Check that

$$\int d^2k\,\Omega_-(\mathbf{k}) = \pi\mu^z\mathrm{sign}(m). \tag{54}$$

To do so, extend the integral over the BZ to infinity and recall $\int_0^\infty dx\, x/(x^2 + b^2)^{3/2} = 1/\sqrt{b^2}$.

$$\begin{aligned}\int d^2k\,\Omega_-(\mathbf{k}) &= \int d^2k\mu^z\frac{mv^2}{2(v^2k^2 + m^2)^{3/2}} &(55)\\ &= \int_0^{2\pi}d\theta\int_0^\infty dk\frac{\mu^z m}{2|v|}\frac{k}{(k^2 + (m/v)^2)^{3/2}} &(56)\\ &= \frac{\pi\mu^z m}{|v|}\left|\frac{v}{m}\right| &(57)\\ &= \pi\mu^z\mathrm{sign}(m). &(58)\end{aligned}$$

9. Note the appearance of μ^z. This appears because the valley determines the sense of winding of the Dirac point, or chirality. For a given sign of mass, opposite chirality gives opposite Berry curvature. The integrand is strongly peaked in a region of width m/v in momentum space around the Dirac point. So when the Fermi level lies in the gap formed by the mass, we can say, using the general formula $C_n = \frac{1}{2\pi}\int_{\mathrm{BZ}} d^2k\Omega_n(\mathbf{k})$, that each Dirac point contributes plus or minus half an integer to the Chern number. This must be added for every distinct Dirac point, i.e. for each spin and valley. Therefore there is a general formula for the Chern number for a set of massive Dirac points with the Fermi level in the gap:

$$C = \sum_i \frac{1}{2}\mathrm{sign}(m_i\mu_i^z) \tag{59}$$

Here the sum is over all Dirac points, i.e. for our model of graphene it includes four such points, for spin and valley. One might be worried about Eq. (59), because it looks like it can give a half-integer quantum Hall effect. However, for any physical two dimensional system, there is a famous theorem (Nielsen-Ninomiya) that there must always be an even number of Dirac points. This guarantees an integer result for an insulator.

10. Let's apply the formula Eq. (59) to two limit cases.

    (a) Apply Eq. (59) in the limit $m_2 = 0$.

In this case, we have the mass m_1, which is the same for both valleys:

$$C^{\mathrm{CDW}} = 2_{\mathrm{spin}} \times \left(\frac{1}{2}\mathrm{sign}(m_1) - \frac{1}{2}\mathrm{sign}(m_1)\right) = 0. \tag{60}$$

As mentioned in class, CDW stands for "charge density wave" and represents a state which has a nonuniform ("wave") charge density because of the two valleys have different chemical potential. As mentioned below, it has zero Hall conductivity because the contributions from the two valleys have opposite sign, as expected since it is time-reversal invariant ($\tau^z \to \tau^z$ under time-reversal).

    (b) Now consider the limit $m_1 = 0$.

Now we have mass m_2 which is opposite for the two valleys. We obtain

$$C^{\mathrm{QAHE}} = 2_{\mathrm{spin}} \times \left(\frac{1}{2}\mathrm{sign}(m_2) + \frac{1}{2}\mathrm{sign}(m_2)\right) = 2\mathrm{sign}(m_2). \tag{61}$$

QAHE stands for "quantum anomalous Hall effect". As mentioned below, this state, which has a non-zero Chern number, in turn has edge states and a nonzero Hall conductivity. This is also consistent with the fact that the m_2 mass term is odd under time-reversal (TR), since $\mu^z\tau^z \to -\mu^z\tau^z$ under TR.

Remark: Can these insulators be distinguished experimentally? Yes! The nontrivial ($C \neq 0$ in this case) insulators have edge states and a nonzero (and quantized) Hall conductivity, which the "trivial" ($C = 0$ here) insulators do not have. In fact, one can show that

$$C = N_R - N_L, \tag{62}$$

where N_R and N_L are the number of "right" and "left" movers respectively at a boundary. This is an instance of the "bulk-boundary correspondence";

$$\sigma_H = \frac{\sigma_{yx} - \sigma_{xy}}{2} = e^2 \sum_{n\,\mathrm{occupied}} \int \frac{d^2k}{(2\pi)^2}\Omega_n(\mathbf{k}) = \frac{e^2}{2\pi} \sum_{n\,\mathrm{occupied}} C_n = \frac{e^2}{h}C, \tag{63}$$

where we restored the factor $\hbar$ to get physical units. This is the TKNN formula. (One can also understand it from the bulk-boundary correspondence.) We have therefore shown that one can have a nonzero Hall conductivity in the absence of a magnetic field.