# ICFP M1 - QUANTUM MATTER - TD n°7 - Exercises
# Spin liquids

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## 1 Quantum spin ice: emergent quantum electrodynamics

The pyrochlore lattice is a lattice of corner-sharing tetrahedra (see Fig. 1B). The underlying Bravais lattice is a Face Centered Cubic (FCC) lattice (see Fig. 1A), and the lattice basis is a regular tetrahedron. The unit cell is made of two tetrahedrons.

1. How many sites per unit cell are there?

   The diamond lattice is a lattice with a two-atom basis (see Fig. 1C). The underlying Bravais lattice is an FCC lattice, and, if a diamond lattice site sits at an FCC site, the second atom (other sublattice) sits a quarter of the way along the diagonal of the FCC cubic cell.

2. What lattice do the centers of the (nearest-neighbor) *bonds* of the diamond lattice form?

3. Where do the *sites* of the diamond lattice sit with respect to the *tetrahedra* of the pyrochlore lattice?

   Notice that

   - there are two types of tetrahedra, the "up-pointing" tetrahedra and the "down-pointing" ones. This is consistent with the diamond lattice, which has *two* sites in its basis, being the dual of the pyrochlore lattice.
   - the lattice decomposes into alternating kagomé (corner-sharing triangles in 2d) and triangular planes in various directions, and most importantly that the kagomé planes contain hexagons (loops of length 6), see Fig. 2.

4. What is known as *classical* spin ice is the antiferromagnetic ($J_z > 0$) Ising model on the pyrochlore lattice. Rewrite the Ising terms on the bonds of a single tetrahedron as the square of a sum (up to a constant).

5. What is the condition which should be fulfilled by the ground state(s) (recall $J_z > 0$)?

6. For $S = 1/2$ find all (degenerate) ground states of a single tetrahedron.

7. If the tetrahedra were truly independent, what would the ground state degeneracy of a lattice of $N$ unit cells be?

   We now add quantum fluctuations, *i.e.* terms in the Hamiltonian which do not commute with the Ising part. In particular, we consider "XY" or "transverse" terms:

$$H_{\pm} = -J_{\pm} \sum_{\langle ij \rangle} (S_i^+ S_j^- \pm S_i^- S_j^+). \tag{1}$$

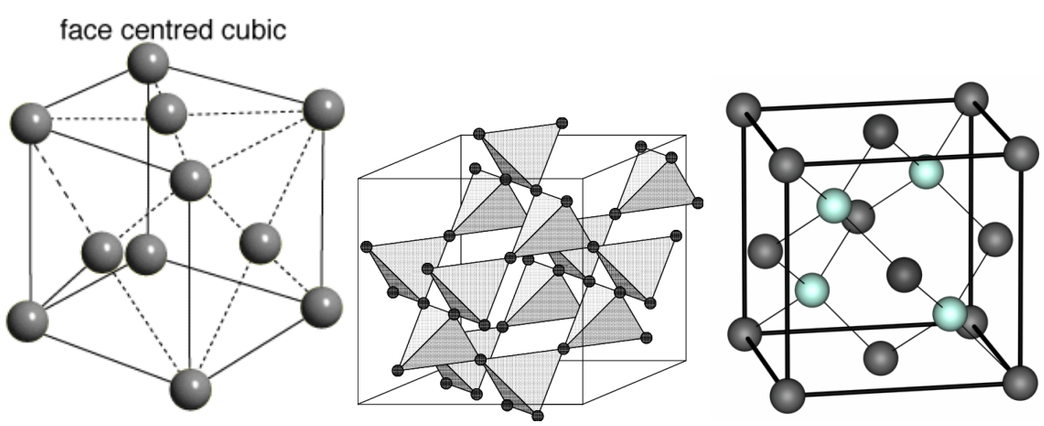


Figure 1: Some crystallographic structures. A: the face centerred cubic lattice. B: The pyrochlore lattice. C: The diamond lattice.

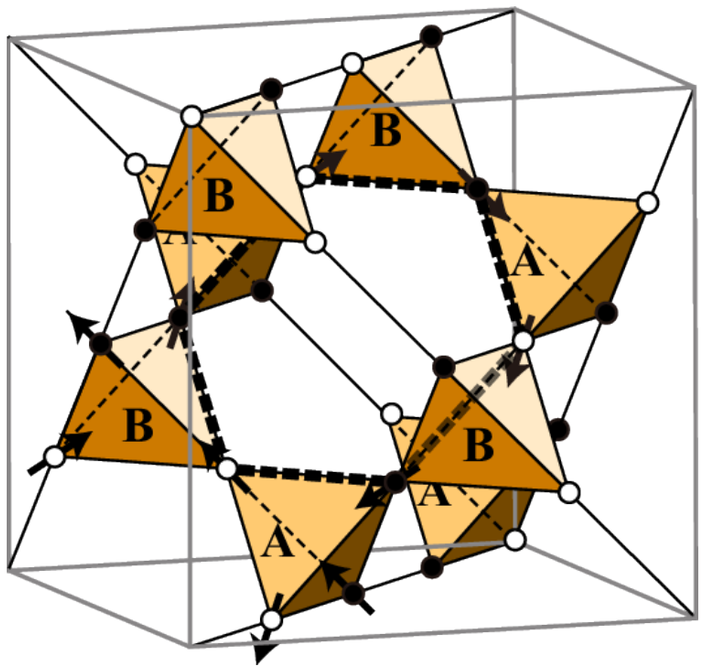


Figure 2: Hexagone loops in the pyrochlore lattice.

The effective Hamiltonian in perturbation theory in small $J_\pm/J_z$ actually vanishes at first and second orders. The Hamiltonian at third order takes the form

$$H_{\text{eff}} = -K \sum_{\{i,j,k,l,m,n\}=\text{hexagon}} \left( S_i^+ S_j^- S_k^+ S_l^- S_m^+ S_n^- + \text{h.c.} \right) \tag{2}$$

8. How does $K$ scale as a function of $J_\pm$ and $J_z$?

9. Show that indeed this Hamiltonian acts purely in the ground state manifold.

   We will now show the relation of this to a $U(1)$ lattice gauge theory. Since the effective Hamiltonian operates within the space with $\sum_{i\in t} S_i^z = 0$, it necessarily commutes with $\sum_{i\in t} S_i^z$ for every tetrahedron $t$ separately.

10. Where does this come from within $H_{\text{eff}}$?

    More formally, we may define

$$Q_t = \epsilon_t \sum_{i\in t} S_i^z \underset{0}{=} 0, \tag{3}$$

where for convenience we included the factor $\epsilon_t$ which assigns a sign that discriminates two orientations of tetrahedra: $\epsilon_t = +1(-1)$ for up (down) oriented tetrahedra. The equality on the right hand side of Eq. (3) holds in the ground state sector (as indicated by the subscript 0 on the equality sign). The "charge" $Q_t$ commutes with $H_{\text{eff}}$. It follows that $H_{\text{eff}}$ is invariant under the $U(1)$ symmetry generated by the unitary operator $U = e^{i\chi Q_t}$, for any $t$ and any $\chi$.

Noting that the charge $Q_t$ is defined on the sites $t$ of a diamond lattice (the tetrahedron centers), we can recast the effective Hamiltonian entirely as a diamond lattice gauge theory. To do so, we define

$$\begin{cases} S^\pm_{t,t'} = S^{\pm\epsilon_t}_{i(t,t')} \\ S^z_{t,t'} = S^z_{i(t,t')} \end{cases}, \tag{4}$$

where $i(t,t')$ is the pyrochlore site shared by neighboring tetrahedra $t$ and $t'$. This definition gives a "vectorial" sense to the operators on the diamond lattice. Then the gauge charge becomes

$$Q_t = \epsilon_t \sum_{t'\in\partial t} S^z_{t,t'} \equiv \epsilon_t (\text{div}[S^z])_t \underset{0}{=} 0, \tag{5}$$

where the sum indicates nearest neighbor sites of $t$, and the result is the lattice divergence at $t$ of the vector field $S^z_{t,t'}$. We recognize therefore $S^z_{t,t'}$ as the lattice analog of the "electric field" in Gauss' law. Let us make things even more explicit by writing things as:

$$S^z_{t,t'} \to E_{t,t'}, \qquad S^\pm_{t,t'} \to e^{\pm i A_{t,t'}}, \tag{6}$$

where $E_{t,t'}$ is a half-integral valued angular momentum, and $A_{t,t'} = -A_{t',t}$ is a $2\pi$-periodic angular variable. If we impose the canonical commutation relations $[E_{t,t'}, A_{t,t'}] = i$ (think of $[L^z, \varphi] = i$), then $S^\pm_{t,t'}$ acts as the desired spin raising/lowering operator, except that it can raise/lower "outside" the physical space with $E_{t,t'} = \pm 1/2$.

11. Rewrite $H_{\text{eff}}$ using Eq. (6). Defining the lattice curl as

$$(\text{curl}[A])_{\text{hexagon}} = \sum_{(t\in\text{hex})=1}^{6} A_{t,t+1} \tag{7}$$

which defines a "magnetic flux" $B$ through a plaquette, show that $H_{\text{eff}}$ takes the form of

$$H_{\text{eff}} = -2K \sum_{\text{hexagon}} \cos(\text{curl}[A]). \tag{8}$$

12. Show that this Hamiltonian has a manifest gauge invariance under local phase rotations by an angle $\chi_t$ on each diamond site, for which

$$S^{\pm}_{t,t'} \to e^{\pm i(\chi_{t'}-\chi_t)} S^{\pm}_{t,t'}. \tag{9}$$

13. Now imposing the spin-1/2 constraint y adding a term $U(E^2 - 1/4)$ (notice that $U \to +\infty$ will project out the unphysical states), and assuming small $B = \text{curl}[A]$, show that the Hamiltonian is that of quantum electrodynamics without matter fields $H \sim E^2 + B^2$.

*Remark:* This describes the low energy theory, without matter fields (without "electrons" and "positrons"). To go further and include those, one instead writes $S^{+}_{r,r'} = \Phi^{\dagger}_t e^{iA_{t,t'}} \Phi_{t'}$ and $S^z$ is unchanged, one obtains spinons ($\Phi$) hopping in the background of fluctuating gauge fields, and the gauge transformation must include also $\Phi_t \to \Phi_t e^{-i\chi_t}$.