TD 10: Harmonic analysis for stochastic dynamics

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The autocorrelation function and the Wiener-Kinchin theorem. For stationary stochastic dynamics, the autocorrelation function is defined as $C_x(t,t') = \langle x(t)x(t')\rangle$, where $\langle\cdot\rangle$ denotes the ensemble average (average over the stationary distribution).

1. What property of C_x is ensured by stationarity? Write C_x as a function of $\tau = t' - t$.

2. Assuming ergodicity, show that the autocorrelation function is an even function.

3. *(Bonus)* Show that for any integrable function g,

$$\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds g(s-t) = T \int_{-T}^{T} g(\tau) d\tau.$$

4. The power spectral density of a signal $x(t)$ is defined as

$$\hat{P}_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \langle |\hat{x}_T(\omega)|^2 \rangle,$$

where $\hat{x}_T(\omega) = \int_{-T/2}^{T/2} x(t)e^{i\omega t}dt$, the Fourier transform of $x(t)$ over the interval $t \in [-T/2, T/2]$. Show that the power spectral density is the Fourier transform of the autocorrelation function:

$$\hat{P}_x(\omega) = \mathcal{F}(C_x(\tau))$$

where $\mathcal{F}$ represent the Fourier transform.

5. How can we get the autocorrelation function from the power spectral density? These relations are known as the Wiener-Kinchin theorem.

Leveraging Fourier transforms to estimate the correlation function. We now leverage the Wiener-Kinchin theorem in example applications

6. Consider the following stochastic dynamics for the velocity of a damped particle,

$$m\dot{v} = -\gamma v + \sigma\eta,$$

where η is a Gaussian uncorrelated random variable $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$, γ is a damping coefficient and σ is the magnitude of the noise. Write down the Fourier transform of the Langevin equation, and estimate the correlation function of v through the Wiener-Kinchin theorem.

7. What is the value of σ that keeps the particle at thermal equilibrium? Compare with the results of TD9.

8. Consider now the Langevin dynamics of a particle confined in a harmonic (quadratic) potential and in contact with a heat bath in equilibrium:

$$m\ddot{x} = -\gamma\dot{x} - kx + \eta(t),$$

Where $\eta(t)$ is Gaussian white noise. What is its equation of motion in phase space? What is the correlation function of the noise in equilibrium?

9. Compute the autocorrelation function of the particle's position using the Wiener-Kinchin theorem, and show that it obeys qualitatively different behaviors in the underdamped and overdamped regimes.

Only when you have finished all the exercises

The Wikipedia Moment. MAX PLANCK (1858-1947).

Planck was born in 1858 in Kiel, Holstein and was baptized with the name of Karl Ernst Ludwig Marx Planck. However, by the age of ten he signed with the name Max and used this for the rest of his life. Planck came from a traditional, intellectual family. His father was a law professor at the University.

The Munich physics professor Philipp von Jolly advised Planck against going into physics, saying, "In this field, almost everything is already discovered, and all that remains is to fill a few holes." Planck replied that he did not wish to discover new things, but only to understand the known fundamentals of the field, and so began his studies in 1874 at the University of Munich. Under Jolly's supervision, Planck performed the only experiments of his scientific career but transferred to theoretical physics.

In 1877, he went to the Friedrich Wilhelms University in Berlin for a year of study with physicists Hermann von Helmholtz and Gustav Kirchhoff. While there he undertook a program of mostly self-study of Clausius's writings, which led him to choose thermodynamics as his field. In October 1878, Planck passed his qualifying exams and in February 1879 defended his dissertation, Über den zweiten Hauptsatz der mechanischen Wärmetheorie (On the second law of thermodynamics). Although he was initially ignored by the academic community, he furthered his work on the field of heat theory and discovered one after another the same thermodynamical formalism as Gibbs without realizing it. Clausius's ideas on entropy occupied a central role in his work.

In April 1885, the University of Kiel appointed Planck as associate professor of theoretical physics. Further work on entropy and its treatment, especially as applied in physical chemistry, followed. He published his Treatise on Thermodynamics in 1897. In 1889, he was named the successor to Kirchhoff's position in Berlin and by 1892 became a full professor.

In March 1887, Planck married Marie Merck, sister of a school fellow. They had four children: Karl, the twins Emma and Grete, and Erwin. After several happy years, in July 1909 Marie Planck died, possibly from tuberculosis. In March 1911 Planck married his second wife, Marga von Hoesslin; in December his fifth child Hermann was born.

In 1894, Planck turned his attention to the problem of black-body radiation. The problem had been stated by Kirchhoff in 1859. The question had been explored experimentally, but no theoretical treatment agreed with experimental values. Wilhelm Wien proposed Wien's law, which correctly predicted the behaviour at high frequencies, but failed at low frequencies. The Rayleigh–Jeans law, another approach to the problem, agreed with experimental results at low frequencies, but created what was later known as the "ultraviolet catastrophe" at high frequencies.

Planck derived the first version of the famous Planck black-body radiation law in 1900, which described the experimentally observed black-body spectrum well. Later, Planck revised this first approach, relying on Boltzmann's statistical interpretation of the second law of thermodynamics as a way of gaining a more fundamental understanding of the principles behind his radiation law. As Planck was deeply suspicious of the implications of such an interpretation of Boltzmann's approach, his recourse to them was, as he later put it, "an act of despair ... I was ready to sacrifice any of my previous convictions about physics". At first Planck considered that quantisation was only "a purely formal assumption"; nowadays this assumption, incompatible with classical physics, is regarded as the birth of quantum physics. In recognition of Planck's fundamental contribution to a new branch of physics, he was awarded the Nobel Prize in Physics for 1918.

As a professor at the Friedrich-Wilhelms-Universität in Berlin, Planck joined the local Physical Society. Thanks to his initiative, the various local Physical Societies of Germany merged in 1898 to form the German Physical Society (Deutsche Physikalische Gesellschaft, DPG); from 1905 to 1909 Planck was the president. In 1905, the three epochal papers by Albert Einstein were published in the journal Annalen der Physik. Planck was among the few who immediately recognized the significance of the special theory of relativity. Thanks to his influence, this theory was soon widely accepted in Germany. Planck also contributed considerably to extend the special theory of relativity.

Einstein's hypothesis of light quanta (photons), based on Heinrich Hertz's 1887 discovery of the photoelectric effect, was initially rejected by Planck. He was unwilling to discard completely Maxwell's theory of electrodynamics. Planck and Nernst, seeking to clarify the increasing number of contradictions, organized the First Solvay Conference (Brussels 1911). At this meeting Einstein was able to convince Planck. Meanwhile, Planck had been appointed dean of Berlin University, whereby it was possible for him to call Einstein to Berlin and establish a new professorship for him (1914). Soon the two scientists became close friends

During the First World War Planck's second son Erwin was taken prisoner by the French in 1914, while his oldest son Karl was killed in action at Verdun. Grete died in 1917 while giving birth to her first child. Her sister died the same way two years later. Planck endured these losses stoically.

When the Nazis came to power in 1933, Planck was 74. He witnessed many Jewish friends and colleagues expelled from their positions and humiliated. Again he tried to "persevere and continue working" and asked scientists who were considering emigration to remain in Germany. He hoped the crisis would abate soon and the political situation would improve. At the end of 1938, the Prussian Academy lost its remaining independence and was taken over by Nazis. Planck protested by resigning his presidency.

In January 1945, Erwin, to whom he had been particularly close, was sentenced to death by the Nazi because of his participation in the failed attempt to assassinate Hitler in July 1944. Erwin was executed on 23 January 1945. The death of his son destroyed much of Planck's will to live. After the end of the war Planck, his second wife, and his son by her were brought to Göttingen, where Planck died on 4 October 1947.