TD 11: Fokker-Planck and Noise-Induced Transitions - Solutions

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Properties of the Wiener process. Consider Langevin dynamics of a particle in a viscous fluid,

$$m\ddot{x} = -\gamma\dot{x} + F(t),$$

where $F(t) = \sigma\gamma\eta(t)$ is a random force, where $\langle\eta(t)\rangle = 0$, $\langle\eta(t)\eta(s)\rangle = 2\delta(t-s)$

1. Show that in the overdamped regime the dynamics reduces to
$$\dot{x} = \sigma\eta(t).$$

2. Writing the fluctuations as $\eta(t)dt = dW_t$ and assuming $x(0) = 0$, show that the solution for $x(t)$ in the overdamped regime is
$$x(t) = \sigma W_t$$
W_t is called the Wiener process and is a fundamental concept in the theory of stochastic processes, and the subscript t in W_t represent the argument of the function.

3. Compute $\langle W_t\rangle$ and $\langle W_t W_s\rangle$.

4. Deduce that $\langle \mathrm{d}W_t^2\rangle = 2\mathrm{d}t$, implying $\mathrm{d}W_t \overset{ms}{=} O(\sqrt{\mathrm{d}t})$.[1] Are the trajectories defined by the Wiener process continuous? Are they differentiable?

Deriving the Fokker-Planck equation from the Kramers-Moyal coefficients. Recall that the Fokker-Planck equation can be written in terms of the Kramers-Moyal coefficients as,

$$\partial_t p(x,t) = -\partial_x[D_1(x,t)p(x,t)] + \partial_x^2[D_2(x,t)p(x,t)],$$

where,

$$D_n(x,t) = \frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle (X_{t+\Delta t} - x)^n\rangle|_{X_t = x}\,.$$

5. Leverage the properties we derived for the Wiener process to show that for a stochastic process,
$$\mathrm{d}X_t = a(X_t,t)\mathrm{d}t + b(X_t,t)\mathrm{d}W_t,$$
$D_1(x,t)$ and $D_2(x,t)$ are given by,
$$D_1(x,t) = a(x,t)$$
$$D_2(x,t) = b^2(x,t)$$

6. Show that for an overdamped particle in a potential, $V(x)$, with $\gamma = 1$ and state-dependent (multiplicative) noise $\sigma(x)$, the Fokker-Planck equation is given by,
$$\partial_t p(x,t) = \partial_x[\partial_x V(x)p(x,t)] + \partial_x^2[D(x)p(x,t)]$$
where $D = \sigma^2$.

7. Write down the current J associated with the Fokker-Planck equation,
$$\partial_t p(x,t) = \partial_x J.$$

8. Considering that there is no flux through the domain boundaries, show that the steady-state distribution is given by,
$$p_{st}(x) = \exp\left\{\int^x -\frac{\partial_{x'}V(x') + \partial_{x'}D(x')}{D(x')}dx'\right\},$$
up to a normalization.

9. Show that for additive noise $\sigma(x) = \sigma$ the steady-state distribution reduces to a Boltzmann distribution. Sketch $p_{st}(x)$ for a double well potential with two distinct noise levels $\sigma_1 > \sigma_2$.

10. For multiplicative noise $\sigma(x)$, identify an effective potential $V_{\text{eff}}(x)$. The most likely states, i.e. the maxima of $p_{st}(x)$ or minima of $V_{\text{eff}}(x)$ are no longer at the minimum of the deterministic potential.

[1] The symbol $X \overset{ms}{=} Y$ means equal in mean square, i.e. $\overline{(X-Y)^2} = 0$ implies $X = Y$ almost surely, i.e. X and Y differ at most on a set of measure zero.

Noise-induced transition in logistic growth model. We will now look at an example in which the existence of fluctuations in the model parameters leads to noise-induced transitions between qualitatively different macroscopic behaviors of the system.

11. Consider a model of population growth in which the density of a population $x \in [0,1]$ grows with a rate $\lambda > 0$, but competition for resources yields a decay with $-x^2$, such that,

$$\dot{x} = \lambda x - x^2.$$

Identify its fixed points and their stability w.r.t λ.

12. Compute the solution $x(t)$. Is it consistent with the stability analysis? What happens if we add noise to $\dot{x}$?

13. Consider now that instead of adding noise to $\dot{x}$, we add small Gaussian and white fluctuations in the growth rate, $\lambda \to \lambda'(t) = \lambda + \sigma\eta(t)$. Is the resulting noise additive or multiplicative? Write down the resulting Fokker-Planck equation.

14. Compute the steady-state distribution for vanishing current at the boundaries. What are the most likely states depending in the noise level? Identify a critical noise level at which a transition happens, and describe the different phases.

Correction

1. In the overdamped regime, $m \ll \gamma$, we have,

$$\frac{m}{\gamma}\ddot{x} = -\dot{x} + \frac{F(t)}{\gamma}$$
$$\dot{x} = \frac{F(t)}{\gamma} = \frac{\sigma\gamma\eta(t)}{\gamma}$$
$$\dot{x} = \sigma\eta(t).$$

2. Solving the equation gives,

$$dx = \sigma\eta(t)dt = \sigma dW_t$$
$$x(t) = \sigma W_t$$

3. Giving the fact that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(s) \rangle = 2\delta(t-s)$, we have,

$$\langle W_t \rangle = \langle \int_0^t dW_t' \rangle = \langle \int_0^t \eta(t')dt' \rangle = \int_0^t \langle \eta(t') \rangle dt' = 0,$$

and,

$$\langle W_t W_s \rangle = \langle \int_0^t dW_t' \int_0^s dW_s' \rangle$$
$$= \int_0^t \int_0^s \langle \eta(t')\eta(s') \rangle dt'ds' = 2\int_0^t \int_0^s \delta(t'-s')dt'ds'$$
$$= 2\int_0^t dt'\theta(s-t')$$
$$\langle W_t W_s \rangle = \begin{cases} 2t, t \leq s \\ 2s, t > s \end{cases} = 2\min(t,s).$$

4. $\langle dW_t^2 \rangle = \langle (W_{t+dt} - W_t)^2 \rangle = \langle W_{t+dt}^2 - 2W_{t+dt}W_t + W_t^2 \rangle = 2(t+dt) - 4t + 2t = 2dt$, which implies $dW_t \overset{ms}{=} O(\sqrt{dt})$. Trajectories define by the Wiener process, $x(t) = \sigma W_t$ are continuous but nowhere differentiable because $\frac{dW_t}{dt} = \mathcal{O}(1/\sqrt{dt})$, which diverges when $dt \to 0$.

5. From the Kramers-Moyal coefficients, we must estimate

$$(X_{t+\Delta t} - x)|_{X_t = x} = \int_t^{t+\Delta t} dX_{t'}\Big|_{X_t=x} = \int_t^{t+\Delta t} a(X_{t'},t')\Big|_{X_t=x} dt' + \int_t^{t+\Delta t} b(X_{t'},t')dW_t'\Big|_{X_t=x}.$$

In the limit of $\Delta t \to 0$ we estimate at first order,

$$\int_t^{t+\Delta t} a(X_{t'},t')dt'\Big|_{X_t=x} = a(x,t)\Delta t.$$

Similarly for $b(X_t', t')$,

$$\int_t^{t+\Delta t} b(X_{t'},t')dW_t'\Big|_{X_t=x} = b(x,t)(W_{t+\Delta t} - W_t).$$

Combining these results we get,

$$(X_{t+\Delta t} - x)|_{X_t=x} = a(x,t)\Delta t + b(x,t)(W_{t+\Delta t} - W_t),$$

and thus,

$$D_1 = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle a(x,t)\Delta t + b(x,t)(W_{t+\Delta t} - W_t)\rangle = a(x,t)$$

since $\langle W_t\rangle = 0$. Similarly for D_2,

$$\begin{aligned}
D_2 &= \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left\langle [a(x,t)\Delta t + b(x,t)(W_{t+\Delta t} - W_t)]^2\right\rangle \\
&= \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left[a^2(x,t)\Delta t^2 + 2a(x,t)b(x,t)\Delta t\langle W_{t+\Delta t} - W_t\rangle + b^2(x,t)\langle W_{t+\Delta t}^2 - 2W_tW_{t+\Delta t} + W_t^2\rangle\right] \\
&= \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left[a^2(x,t)\Delta t^2 + b^2(x,t)2\Delta t\right] \\
D_2 &= b^2(x,t).
\end{aligned}$$

6. For an overdamped particle in a potential $V(x)$, the equivalent stochastic differential equation is given by,

$$dx_t = -\partial_x V(x)dt + \sigma(x)dW_t.$$

Therefore, $D_1(x,t) = -\partial_x V(x)$ and $D_2(x,t) = \sigma^2(x) = D(x)$ and so,

$$\partial_t p(x,t) = \partial_x[\partial_x V(x)p(x,t)] + \partial_x^2[D(x)p(x,t)].$$

7. $J = p(x,t)\partial_x V(x) + \partial_x\left[D(x)p(x,t)\right]$

8. The steady-state is given by $\partial_x J = 0 \Rightarrow J =$ constant, since $\partial_t p_{st} = 0$. Since there is no flux through the domain boundaries, that means that $J(x) = 0$ everywhere, and as such,

$$\begin{aligned}
p_{st}(x)\partial_x V(x) + \partial_x\left[D(x)p_{st}(x)\right] &= 0 \\
p_{st}(x)\partial_x V(x) + p_{st}(x)\partial_x D(x) + D(x)\partial_x p_{st}(x) &= 0 \\
\partial_x \log p_{st}(x) &= -\frac{\partial_x D(x) + \partial_x V(x)}{D(x)} \\
p_{st}(x) &= \exp\left\{\int^x -\frac{\partial_{x'}V(x') + \partial_{x'}D(x')}{D(x')}dx'\right\}
\end{aligned}$$

9. For additive noise, we'll have $D(x) = D$ and so

$$\begin{aligned}
p_{st}(x) &= \exp\left\{\int^x -\frac{\partial_{x'}V(x)}{D}\right\} \\
&= e^{-\frac{V(x)}{D}},
\end{aligned}$$

which has the form of a Boltzmann distribution $p_B(x) \propto e^{-\beta V(x)}$, where we identify $\beta^{-1} = D$.

10. For multiplicative noise $\partial_x D(x) \neq 0$, and as such we identify an effective potential as,

$$V_{\text{eff}}(x) = -\int^x \frac{\partial_{x'}V(x') + \partial_{x'}D(x')}{D(x')}dx'.$$

11. To identify the fixed points we compute $\dot{x} = F(x^*) = 0$,

$$\begin{aligned}
\lambda x - x^2 &= 0 \\
x^* = 0 \bigvee x^* &= \lambda.
\end{aligned}$$

To assess the stability of the fixed points we compute the local Jacobian, $\partial_x F|_{x=x^*}$, which yields, $\partial_x F(x^* = 0) = \lambda$,$\partial_x F(x^* = \lambda) = -\lambda$, and so $x^* = \lambda$ is stable while $x^* = 0$ is unstable.

12. Separation of variables yields,

$$\begin{aligned}
\frac{dx}{x(\lambda - x)} &= dt \\
\frac{dx}{\lambda x} + \frac{dx}{\lambda(\lambda - x)} &= dt \\
\frac{dx}{x} + \frac{dx}{\lambda - x} &= \lambda dt.
\end{aligned}$$

Integrating each term yields,

$$\begin{aligned}\log(x) + c_1 - \log(\lambda - x) + c_2 &= \lambda t + c_3 \\ \log\left(\frac{x}{\lambda - x}\right) &= t + c \\ \frac{x}{\lambda - x} &= ce^{\lambda t} \\ x(t) &= \frac{ce^{\lambda t}\lambda}{1 + ce^{\lambda t}}.\end{aligned}$$

Where obviously the dummy constant c changes throughout the derivation. Imposing an initial condition $x(0) = x_0$ we arrive at,

$$x(t) = \frac{x_0 e^{\lambda t}}{1 + \frac{x_0}{\lambda}\left(e^{\lambda t} - 1\right)} = \frac{x_0}{e^{-\lambda t} + \frac{x_0}{\lambda}\left(1 - e^{-\lambda_t}\right)}.$$

Taking $t \to \infty$ yields $x(t \to \infty) = \lambda$, which agrees with the stability analysis. Adding noise would simply result in fluctuations around the deterministic trajectory, such that the most likely state would still be $x^* = \lambda$.

13. Changing λ in the equations of motion yields,

$$\dot{x} = \lambda x - x^2 + \sigma x \eta(t),$$

and thus the noise is clearly multiplicative. From this we can see that, $D_1(x) = \lambda x - x^2$ and $D_2(x) = \sigma^2 x^2$ and so the resulting FP equation is,

$$\partial_t p(x,t) = -\partial_x\left[(\lambda x - x^2)p(x,t)\right] + \partial_x^2\left[\sigma^2 x^2 p(x,t)\right].$$

14. To get the steady-state distribution we can write down the dynamics in terms of a potential $V(x) = -\frac{\lambda x^2}{2} + \frac{x^3}{3}$, and so using the solution to problem 8. we have that the steady-state distribution is given by,

$$\begin{aligned}p_{st}(x) &= \exp\left\{\int^x \frac{\lambda x' - x'^2 - 2\sigma^2 x'}{\sigma^2 x'^2}\right\} \\ &= \exp\left\{\int^x \left(\frac{\lambda}{\sigma^2} - 2\right)\frac{1}{x'}dx' - \int^x \frac{1}{\sigma^2}dx'\right\} \\ &= \exp\left\{\left(\frac{\lambda}{\sigma^2} - 2\right)\log x - \frac{x}{\sigma^2}\right\} \\ p_{st}(x) &= x^{\frac{\lambda}{\sigma^2} - 2} e^{-\frac{x}{\sigma^2}}.\end{aligned}$$

When $\frac{\lambda}{\sigma^2} < 2$, $x = 0$ is a singularity in the density and thus it is the most likely state. Thus, when the noise level $\sigma > \sqrt{\frac{\lambda}{2}}$, the noise destabilizes the $x^* = \lambda$ fixed point, and the dynamics flows to $x^* = 0$. Only when $\lambda \geq 2\sigma^2$ the deterministic dynamics dominates and the stable fixed point according becomes the most likely. This makes sense since the multiplicative nature of the noise (dependence on x) makes it such that the larger x is the larger is the magnitude of the noise, and as such the fluctuations are lower when $x \ll 1$, effectively trapping the system close to 0. From the population dynamics point of view, this tells us that if the environment is such that the fluctuations in growth rate are large, the most likely fate of the population is to go extinct. An illustration of these two regimes can be seen in Fig. 1 Thus, at $\sigma_c = \sqrt{\frac{\lambda}{2}}$ we have a transition between two qualitatively different "phases" of the system. When there is no noise, the system obeys the deterministic dynamics and thus the steady-state invariant distribution is $p_{st}(x) = \delta(x - \lambda)$. In this sense, the observed transition is a purely noise-driven phenomenon, and it's in that sense that we designate such transitions **noise-induced transitions**.

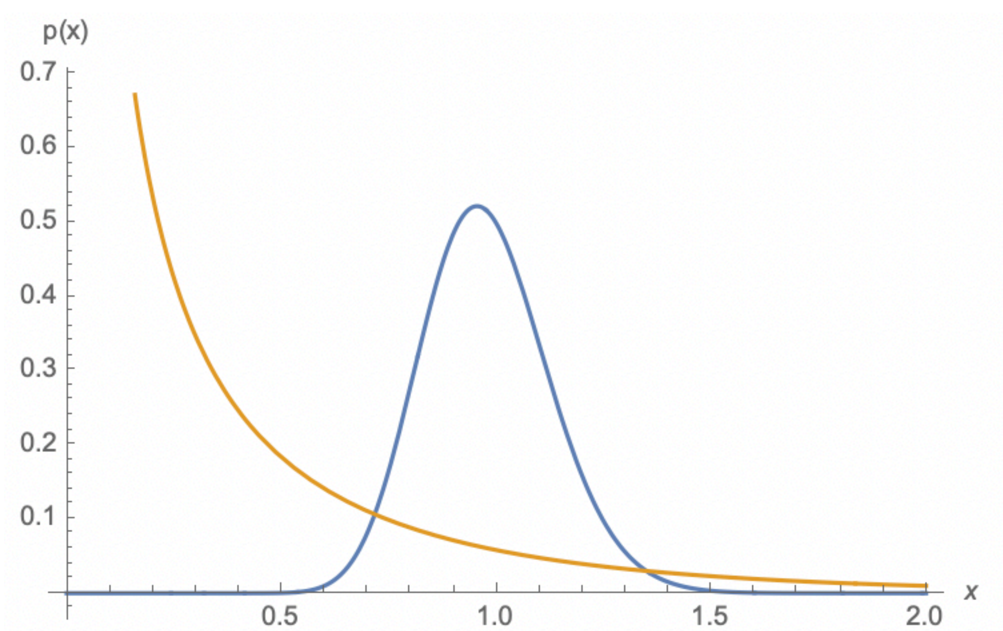

Figure 1: Non-normalized stationary distributions for $\sigma > \sqrt{\lambda/2}$ (orange) and $\sigma < \sqrt{\lambda/2}$ (blue). Here x is not normalized to the interval $[0, 1]$, but the qualitative behavior is the same.