TD 12: First-Passage Problems

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9 décembre 2021

An important quantity when studying stochastic processes, is to know the amount of time it takes before our fluctuating variables reaches a certain threshold. This is one of the simplest observations we can make from an experimental point of view, yet it captures some of properties of the underlying dynamics. The quantity of interest is the *first-passage time*, which is the average time to escape from an interval. It is in itself a stochastic variable with an associated probability distribution. As we will see, finding an expression for the full first-passage time distribution is very challenging except for a few simple systems. However, we can leverage the nature of the Fokker-Planck equation to compute moments of this distribution.

How can we obtain the mean first passage time $\overline{\tau}$?

1. Consider a point-like random walker starting at an initial condition $x \in [0, b]$ at time $t = 0$, jumping $\delta \ll 1$ to the right/left with equal probability after every time step $\Delta t \ll 1$. Show that the mean first-passage time $\overline{\tau}$ satisfies
$$\frac{d^2}{dx^2}\overline{\tau}(x) + \frac{1}{D} = 0,$$
where the diffusion constant $D = \frac{\delta^2}{2\Delta t}$.

2. Assume that the boundaries of the interval are absorbing. What does this imply for the boundary conditions of $\overline{\tau(x)}$ at $x = 0$ and $x = b$? Solve for $\overline{\tau}(x)$ with absorbing boundaries.

3. What is the average of $\overline{\tau}(x)$ if x is chosen uniformly at random in $[0, b]$?

Forward and backward Fokker-Planck equations and the mean-first passage time. Here, we derive the general expression of the mean-first passage time for a one-dimensional stochastic process driven by thermal noise

4. The propagator of the density dynamics of a system driven by thermal noise obeys the Fokker-Planck equation,
$$\partial_t p(x,t|x',t') = [-\partial_x D_1(x,t) + \partial_x^2 D_2(x,t)]p(x,t|x',t') \equiv \mathcal{L}p(x,t|x',t').$$
Multiply by $p(x',t'|x,t)$ and integrate by parts (assuming that probability vanishes sufficiently fast at $\pm\infty$) to derive the *adjoint* (also known as *backward*) Fokker-Planck equation,
$$-\partial_{t'} p(x,t|x',t') = [D_1(x',t')\partial_{x'} + D_2(x',t')\partial_{x'}^2]p(x,t|x',t') \equiv \mathcal{L}^+ p(x,t|x',t').$$

5. Consider the same situation as before but for general 1D Fokker-Planck dynamics with arbitrary drift and diffusion. We define the *survival probability* as the probability that the particle is still in the interval $[a, b]$ after a time t as
$$S(t|x) \equiv \int_a^b p(x',t|x,0)\mathrm{d}x'.$$
With absorbing boundaries $S(t|x)$ decreases in time such that $S(t|x) \overset{t\to\infty}{\to} 0$. The rate at which the probability decreases, i.e. the rate at which the particles in our ensemble reach the absorbing boundary, is given
$$f(t|x) = -\partial_t S(t|x).$$
Express the mean first-passage time $\overline{\tau}(x)$ using f and, assuming $tS(t|x) \overset{t\to\infty}{\to} 0$, simplify to show that
$$\overline{\tau}(x) = \int_0^\infty S(t|x)\mathrm{d}t.$$

6. The function $S(t|x)$ satisfies the initial condition $S(0|x) = 1 \forall x \in (a, b)$. Assume a stationary random process (D_1, D_2 do not depend on time), such that $p(x',t|x,t_0) = p(x',t-t_0|x,0)$. Apply $\mathcal{L}^+$ to $\overline{\tau}$ and show that
$$\mathcal{L}^+\overline{\tau}(x') = -1 \tag{1}$$
Check that that 1 agrees with problem 1. for $D_1 = 0, D_2 = D = const.$[1]

[1] One can derive an equation analogous to (1) for all higher moments of the first passage time distribution. In the simple case of $D_1 = 0$, $D_2 = D = cst$, one may even compute f exactly. It turns out to be a fairly wide distribution with a standard deviation comparable to the mean. This illustrates that the *mean* first passage time alone does not contain all information, but it is often the only thing one may hope to obtain analytically.

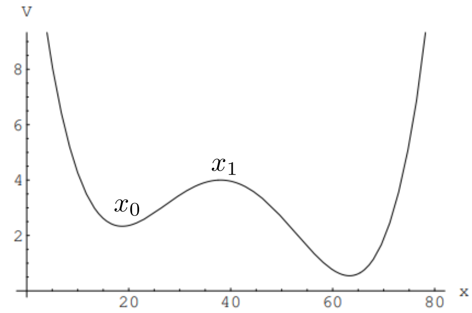

Figure 1: Sketch of the double-well potential $V(x)$ considered here.

7. *(Bonus)* Assume a to be reflecting, i.e. $\partial_x S(t|x)|_{x=a} = 0$, while b is absorbing. Show that the general solution to eq.(1) is

$$\overline{\tau}(x) = \int_x^b \mathrm{d}z \frac{1}{\phi(z)} \int_a^z \frac{\phi(y)}{D_2(y)} dy,$$

where

$$\phi(y) \equiv \exp\left(\int^y \mathrm{d}x \frac{D_1(x)}{D_2(x)} \right).$$

Example applications. We will compute the mean first passage times for some example dynamics.

8. Consider the case of $D_1(x) = -\lambda x$, $\lambda > 0$, $D_2(x) = D = const.$. It describes the dynamics of an overdamped particle in a quadratic potential subject to a space-independent random force, and is known as the *Ornstein Uhlenbeck-Process*. Let $a = 0$. Use the asymptotics $\int_0^x e^{t^2} dt \sim \frac{e^{x^2}}{2x}$ $(x \to \infty)$ to show that at leading order in $\eta = \lambda b^2/(2D) \gg 1$ (what does this limit mean physically?)

$$\overline{\tau}(x=0) \sim \frac{\sqrt{\pi}}{2\lambda} \frac{e^{\eta^2}}{\eta}$$

Interpret this result.

9. Consider a double-well potential $V(x)$ as depicted in figure 1, with $D_1(x) = -\frac{dV}{dx}$. Let the starting point be x_0 at $t = 0$. We want know the mean first passage time for the escape, due to additive noise, over the potential barrier into the right potential well. Let the maximum be located at x_1. We choose $a = -\infty$ reflective and b lying somewhat to the right of x_1, absorbing. Consider appropriately small noise $D_2 = D$ and use the "saddle point" approximation, for a function $f(x)$ whose local maximum is at $x = x_*$, and a large parameter $\Lambda \to \infty$,

$$\int e^{\Lambda f(x)} dx \sim \sqrt{\frac{2\pi}{\Lambda |f''(x_*)|}} e^{\Lambda f(x_*)}, \tag{2}$$

to show that at leading order

$$\overline{\tau}(x_0) \approx \frac{2\pi}{\sqrt{V''(x_0)|V''(x_1)|}} e^{(V(x_1)-V(x_0))/D}.$$

——————— *Only when you have finished all the exercises* ———————

The Wikipedia Moment. ADRIAAN FOKKER (1887-1972).

Adriaan Daniël Fokker was born on 17 August 1887 in Buitenzorg, Dutch East Indies (now Bogor, Indonesia). Fokker studied mining engineering at the Delft University of Technology and physics at the University of Leiden with Hendrik Lorentz, where he earned his doctorate in 1913. In his 1913 thesis, he derived the Fokker–Planck equation along with Max Planck. After his military service during World War I he returned to Leiden as Lorentz' and Ehrenfest's assistant. Fokker became a physics teacher at the Gymnasium of Delft after 1918 and was appointed in 1923 as the first professor of Applied Physics at the Technische Hoogeschool Delft (today Delft University of Technology).

Fokker made several contributions to special relativity, and some less well-known contributions to general relativity, particularly in the area of geodetic precession, the phenomena of precession of a freely falling gyroscope in a gravitational field.

Fokker began to study music theory during the Second World War, when the University of Leiden was closed; partly this was due to a desire to convince the Nazis he would be of no use to the war effort, and partly it was a response to reading the work of Christiaan Huygens on the 31 equal temperament. In 1949 he became member of the Royal Netherlands Academy of Arts and Sciences. He died on 24 September 1972, at the age of 85, in Beekbergen near Apeldoorn.