TD 15: Revision - Solutions

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Consider the dynamics of an overdamped particle influenced by a fluctuating external parameter Λ_t,

$$\dot{x} = F_\Lambda(x) = f(x) + \Lambda_t g(x). \tag{1}$$

Assume that the fluctuations in Λ are fast enough to be modelled as Gaussian white noise, $\Lambda_t = \lambda + \sigma\eta_t$, with $\langle \eta_t \rangle = 0$ and $\langle \eta_t \eta_s \rangle = 2\delta(t-s)$.

1. Identify the deterministic and stochastic components. Is the noise additive or multiplicative?

2. Use the Kramers-Moyal coefficients to deduce the corresponding Fokker-Planck equation,

$$\partial_t \rho(y,t) = -\partial_y \left[F_\lambda(y)\rho(y,t)\right] + \sigma^2 \partial_y^2 \left[g^2(y)\rho(y,t)\right], \tag{2}$$

   identifying F_λ.

3. Estimate the stationary distribution ρ_s for a system with natural boundary conditions (vanishing probability current), up to a normalization constant N,

$$\rho_s(x) = N g^{-2}(x) \exp\left[\frac{1}{\sigma^2}\int^x \frac{F_\lambda(u)}{g^2(u)} du\right]. \tag{3}$$

4. Using the previous notation, and $\beta = 1/\sigma^2$, identify an effective potential V_{eff} corresponding to the stationary distribution.

5. Analogous to fixed points in a deterministic dynamical system, the maxima of the stationary distribution correspond to regions of state space where the system lingers more often. Do these maxima correspond to the fixed points of the deterministic system (for which $\sigma = 0$)? Explain while comparing the effects of additive and multiplicative noise.

Population genetics. Consider now the following deterministic model for the fluctuation of the population of an allele in a genetic population, x,

$$\dot{x} = \frac{1}{2} - x + \Lambda x(1-x),\ x \in [0,1], \tag{4}$$

where the rate Λ is related to the selectivity of the allele.

6. Identify the physically relevant fixed point and assess their stability.

7. Imagine now that the environment induces fluctuations in the rate of selection of the allele such that $\Lambda \to \Lambda_t = \lambda + \sigma\eta_t$, where $\langle \eta_t \rangle = 0$, $\langle \eta_t \eta_s \rangle = 2\delta(t-s)$ is Gaussian white noise. Write down the stationary distribution for a vanishing probability current (the final result may contain an unsolved integral).

8. Taking $\lambda = 0$ for simplicity, identify an effective potential and locate the maxima of the stationary distribution.

9. Draw $\rho_s(x)$ for different values of σ, discuss how the noise amplitude results in macroscopically different phases of the system and identify the critical noise level. How is this different from the noiseless case ($\sigma = 0$)?

10. (Bonus) What happens when $|\lambda| > 0$?

Correction

1. Plugging in $\Lambda_t = \lambda + \sigma\eta_t$, we get,

$$\dot{x} = f(x) + \lambda g(x) + \sigma g(x)\eta_t$$
$$\dot{x} = F_\lambda(x) + \sigma g(x)\eta_t.$$

   The deterministic component is $F_\lambda(x) = f(x) + \lambda g(x)$ and the stochastic component is $\sigma g(x)\eta_t$. Since the stochastic term depends on x, the noise is **multiplicative**.

2. From TD11, we have

$$D_1(x,t) = a(x,t)$$
$$D_2(x,t) = b^2(x,t),$$

where $a(x,t)$ corresponds to the deterministic term, while $b(x,t)$ is the stochastic term. Therefore, in our notation we have

$$D_1(x,t) = F_\lambda(x)$$
$$D_2(x,t) = \sigma^2 g^2(x).$$

The corresponding Fokker-Planck equation is thus

$$\partial_t \rho(x,t) = -\partial_x \left[F_\lambda(x)\rho(x,t)\right] + \sigma^2\partial_x^2\left[g^2(x)\rho(x,t)\right].$$

3. The natural boundary conditions mean that the current J is vanishing. We identify the current by writing,

$$\partial_t \rho(x,t) = \partial_x J(x,t),$$

where

$$J = -F_\lambda(x)\rho(x,t) + \sigma^2\partial_x g^2(x)\rho(x,t).$$

To obtain the stationary distribution $\rho_s(x)$, we take $J = 0$:

$$-F_\lambda(x)\rho_s(x) + \sigma^2\partial_x g^2(x)\rho_s(x) = 0$$
$$2g(x)\partial_x g(x)\rho_s(x) + g^2(x)\partial_x\rho_s(x) = \frac{F_\lambda(x)\rho_s(x)}{\sigma^2}$$
$$\partial_x \log\rho_s(x) = \frac{F_\lambda(x)}{\sigma^2 g^2(x)} - 2\partial_x \log g(x)$$
$$\rho_s(x) = g^{-2}(x)\exp\left\{\frac{1}{\sigma^2}\int^x \frac{F_\lambda(u)}{g^2(u)}du\right\}$$

The normalization condition $\int \rho_s(x)dx = 1$ yields a normalization constant N such that,

$$\rho_s(x) = Ng^{-2}(x)\exp\left\{\frac{1}{\sigma^2}\int^x \frac{F_\lambda(u)}{g^2(u)}du\right\}.$$

Note that in equilibrium with a heat bath $N = 1/Z$ where Z is the partition function of the canonical ensemble.

4. To identify an effective potential, we write the stationary distribution in the Boltzmann form,

$$\rho_s(x) = \frac{1}{Z}e^{-\beta V} = \frac{1}{Z}\exp\left[\frac{1}{\sigma^2}\int^x \frac{F_\lambda(u) - 2\sigma^2 g(u)\partial_u g(u)}{g^2(u)}du\right].$$

With $\beta = 1/\sigma^2$ we identify,

$$V_{\text{eff}} = \int^x \frac{F_\lambda(u) - 2\sigma^2 g(u)\partial_u g(u)}{g^2(u)}du$$

.

5. The maxima of the stationary distribution correspond to the minima of the effective potential, which are the roots of,

$$F_\lambda(x^*) - 2\sigma^2 g(x^*)\partial_x g(x)|_{x^*} = 0$$

The deterministic system is $\dot{x} = F_\lambda(x)$ and so its fixed points are the roots of $F_\lambda(x^*)$. In this sense, the maxima of the stationary distribution are only equal to those of the deterministic system if the noise is additive, in which case $\partial_x g(x) = 0$.

6. The fixed points correspond to $\dot{x} = 0$,

$$\frac{1}{2} - x^* + \Lambda x^*(x^* - 1) = 0$$
$$x^* = \frac{(\Lambda - 1) \pm \sqrt{\Lambda^2 + 1}}{2\Lambda}.$$

The only physically relevant fixed point in the domain where x is defined, $x \in [0,1]$ is the (+) solution,

$$x^* = \frac{(\Lambda - 1) + \sqrt{\Lambda^2 + 1}}{2\Lambda}.$$

To assess stability, we can compute $\partial_x \dot{x}|_{x^*} = -\sqrt{\Lambda^2 + 1} < 0$, so the fixed point is asymptotically stable for all Λ.

7. Plugging Λ with $\lambda + \sigma\eta_t$ we get,

$$\dot{x} = \frac{1}{2} - x + \lambda x(1-x) + \sigma x(1-x)\eta_t$$

Identifying $F_\lambda(x) = \frac{1}{2} - x + \lambda x(1-x)$ and $g(x) = x(1-x)$, we get the following stationary distribution,

$$\rho_s(x) = N x^{-2}(1-x)^{-2} \exp\left\{ \frac{1}{\sigma^2} \int^x \frac{\frac{1}{2} - u + \lambda u(1-u)}{u^2(1-u)^2} du \right\}$$

8. With $\lambda = 0$, the effective potential can be written as,

$$V_{\text{eff}}(x) = \int^x \frac{1/2 - u - 2\sigma^2 u(1-u)(1-2u)}{u^2(1-u)^2} du.$$

The maxima of the stationary distribution are given by the minima of the potential $\hat{x}$, which are obtained through,

$$\partial_x V_{\text{eff}}\big|_{x=\hat{x}} = \frac{1/2 - \hat{x} - 2\sigma^2 \hat{x}(1-\hat{x})(1-2\hat{x})}{\hat{x}^2(1-\hat{x})^2} = 0.$$

There are three roots, which can be identified by factorizing $1/2 - \hat{x}$,

$$\left(\frac{1}{2} - \hat{x}\right) - 4\sigma^2 \hat{x}(1-\hat{x})\left(\frac{1}{2} - \hat{x}\right) = 0$$
$$\left(\frac{1}{2} - \hat{x}\right)\left[1 - 4\sigma^2 \hat{x}(1-\hat{x})\right] = 0$$
$$\hat{x} = \frac{1}{2} \vee \hat{x} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{\sigma^2}}\right).$$

9. For V_{eff} to have three real roots, we need $\sigma > 1$. As we'll show, in this case the stationary distribution has two peaks at $\hat{x} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{\sigma^2}}\right)$ and a minimum at $\hat{x} = 1/2$. To show this, we compute the second derivative of V_{eff} at the fixed points. Writing an auxiliary function $a(x) = 1/2 - x - 2\sigma^2 x(1-x)(1-2x)$, we have

$$\begin{aligned} \partial_x^2 V_{\text{eff}}\big|_{x=\hat{x}} &= \left.\frac{\partial_x a(x)}{x^2(1-x)^2}\right|_{x=\hat{x}} + a(x)\partial_x\left[x^{-2}(1-x)^{-2}\right]\big|_{x=\hat{x}} \\ &= \left.\frac{\partial_x a(x)}{x^2(1-x)^2}\right|_{x=\hat{x}}, \end{aligned}$$

Since $a(\hat{x}) = 0$. Therefore, the sign of $\partial_x^2 V_{\text{eff}}\big|_{x=\hat{x}}$ and thus the stability of the fixed points depends solely on the sign of $\partial_x a(x)$ at $\hat{x}$:

$$\partial_x a\big|_{x=\hat{x}} = -1 - 2\sigma^2\left[1 + 6(\hat{x}^2 - \hat{x})\right]$$

For $\hat{x} = 1/2$, we have $\partial_x a(x)|_{x=1/2} = \sigma^2 - 1$, which is positive (unstable) when $\sigma > 1$ and negative (stable) when $\sigma < 1$. For $\hat{x}_M = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{1}{\sigma^2}}\right)$ we have $\hat{x}^2 - \hat{x} = -\frac{1}{4\sigma^2}$ and thus $\partial_x a(x)|_{x=x_M} = 2(1-\sigma^2)$, which is unstable when $\sigma < 1$ and stable when $\sigma > 1$. Thus, at $\sigma > 1$ we have two stable peaks at $\hat{x}_M$ and a minimum at $\hat{x} = 1/2$. Lowering σ to below a critical noise level $\sigma_c = 1/2$, the two stable fixed points collapse and the fixed point at $x = 1/2$ becomes the only stable fixed point. These qualitatively different phases of the system are depicted in Fig. 1. As we have seen in problem 6., in the absence of noise we have only one stable fixed point. Therefore, the resulting macroscopic change in behavior is purely driven by noise: it is a **noise-induced transition**.

10. Below the critical noise level, for $\lambda > 0$ ($\lambda < 0$) the deterministic fixed point will move towards $x = 1$ ($x = 0$). However for large enough σ a noise-induced transition is still observed and the stationary distribution becomes bimodal above a certain $\sigma_c(\lambda)$ which now depends on λ. This can be seen by plotting the extrema of the stationary distribution ρ_s^* for different values of σ and as a function of λ, Fig. 2a. If we keep $\sigma > 1$ and we vary λ, the behavior resembles that of a first order phase transition, as is clear from the sigmoidal form of the curve for the extrema of ρ_s. This shows that $\sigma_c = 1$ is indeed a critical variance above which hysteresis phenomenon for the extrema occurs: we have a cusp catastrophe in the (λ, σ) plane, with a critical point at $(0, 1)$, Fig. 2b. Another way to visualize this qualitative change is to consider the shape of the effective potential. Below the critical noise, the potential has only one well with a minimum at the deterministic fixed point, and the effect of the noise is simply to broaden the distribution. Above the critical noise, this well weaves up and two new wells emerge since the boundaries at $x = 0$ and $x = 1$ remain natural boundaries. This shows that in contrast with additive noise, the multiplicative noise not only "disorganizes" the system, but it may also stabilize macroscopic states.

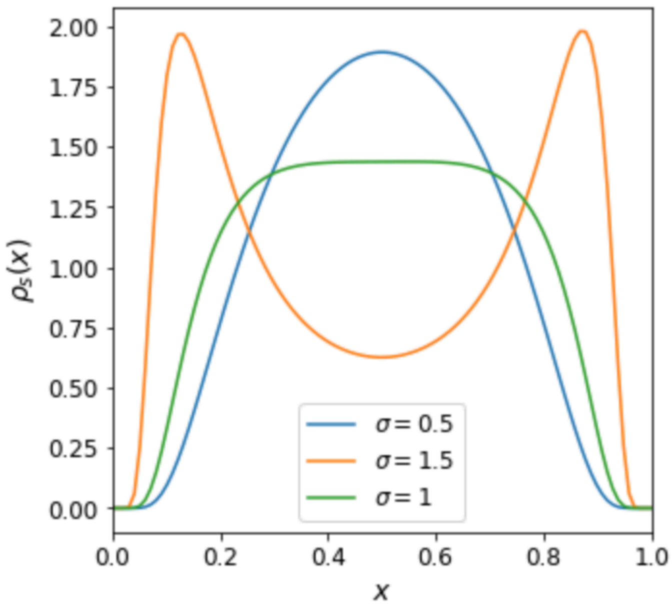
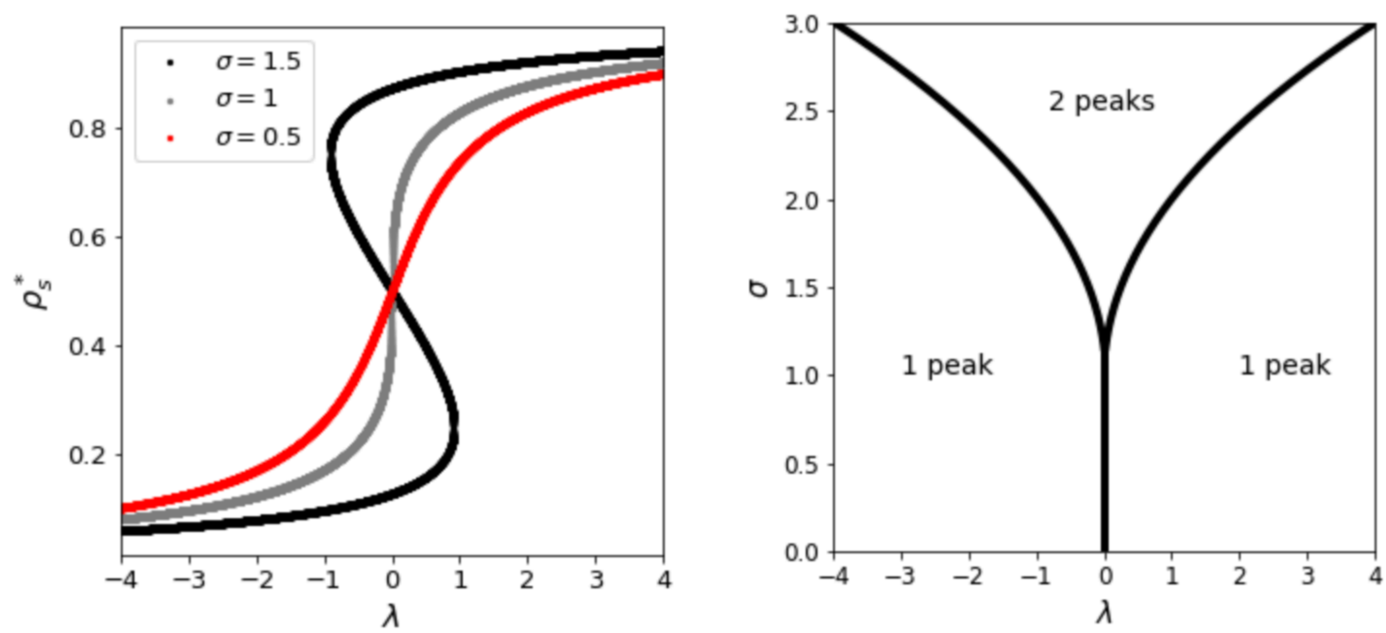


Figure 1: Stationary distribution for different values of σ.



(a) Extrema of ρ_s as a function of λ.

(b) Phase diagram exhibiting cusp catastrophe.

Figure 2: Extrema of ρ_s and phase diagram as a function of (λ, σ).