# TD 2: From Heat Engines to Tropical Cyclones - Solutions

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Every years, there about 80 cyclones in world, the majority in the Pacific ocean and about 20 in the Atlantic ocean. Formed in the inter-tropical areas , they can be dangerous when they touch the land. Since 30 years, the cyclones are modelled as a heat machine working thanks the difference of temperature between the ocean and the troposphere. This exercise is based on the work of Kerry Emanuel (https://emanuel.mit.edu/).

The model is a presented in the scheme. The ocean is a thermal reservoir at $T_c = 300$ K, while the troposphere is a thermal reservoir at $T_f = 200$ K. The system is is made of a mass $m_a$ of dry air and $\delta m_w$ of water, which is a liquid or a gas depending on the steps of the process. The typical size of the system is $L = 10$ m. All the transformations are supposed reversible and we denote $P_i, v_i, z_i$ the pressure, velocity and height at the point $i \in \{A, B, C, D\}$. We remind that the massive enthalpy of water vaporisation is $l_v = 2500\,\text{kJ} \cdot \text{kg}^{-1}$.

1. In a cyclic process, what physical quantity should be study? Write the variation of this quantity, in particular in function of the effective work $W'$.

2. Remind the thermodynamic identity of this physical quantity and the link between the variation of entropy $S$ and the heat $Q$.

**Process from $A$ to $B$.** The air is $A$ is dry and moves to the cyclone's eye where the pressure is low. The interaction with the ocean have two consequences. First, the temperature is kept at $T_c$. Second, the mass $\delta m_w$ gets vaporised and mixes with the air. At this point, $\delta m_w / m_a \approx 5 \cdot 10^{-3} \ll 1$.

3. Express the variation of enthalpy $\Delta H_{AB}$, entropy $\Delta S_{AB}$, the effective work $W'_{AB}$ and the heat $Q_{AB}$. We use the approximation $v_A \ll v_B$.

4. We will see that $W'_{AB} < 0$, what does it represent physically?

**Process from $B$ to $C$.** The air is in the cyclone's eye and moves up adiabatically. During this ascension, the water becomes liquid and falls in the ocean (it's raining). At the point $C$, the air has reached the temperature $T_f$ and is close to mechanical equilibirum, *i.e.* $v_c \ll v_B$. There is no work during this process.

5. Give orders of magnitude of the ascension time, the diffusion time and the mechanical time to justify that the process is adiabatic.

6. Express the variation of enthalpy $\Delta H_{BC}$, entropy $\Delta S_{BC}$, the effective work $W'_{BC}$ and the heat $Q_{BC}$.

7. Calculate the height $z_C$. What is the role of the vapour of water?

**Process from $C$ to $D$.** The air is in the troposphere at $T_f$ and transfer heat to the space by radiation. Its height decreases to $z_D < z_C$. Since $v_B$ is the dominant velocity, we may set $v_C = v_D$.

8. Express the change of pressure as a function of the change of height.

9. Express the variation of enthalpy $\Delta H_{CD}$, entropy $\Delta S_{CD}$, the effective work $W'_{CD}$ and the heat $Q_{CD}$.

**Process from $D$ to $A$.** The air is going back to its initial state. Again, this process is adiabatic.

10. Express the variation of enthalpy $\Delta H_{DA}$, entropy $\Delta S_{DA}$, the effective work $W'_{DA}$ and the heat $Q_{DA}$.

11. Deduce $z_D - z_C$.

**Conclusion.** We now make global balance equations.

12. Express $v_B$. We may introduce $\eta$ the Carnot yield.

13. Given that for a large hurricane, the depression at the eye may be of the order $P_B \sim 10^4$ Pa, give an estimate of $v_B$.

14. We define the yield of the cyclone as $\eta^* = -\frac{W'}{Q_{AB}}$. Give its expression.

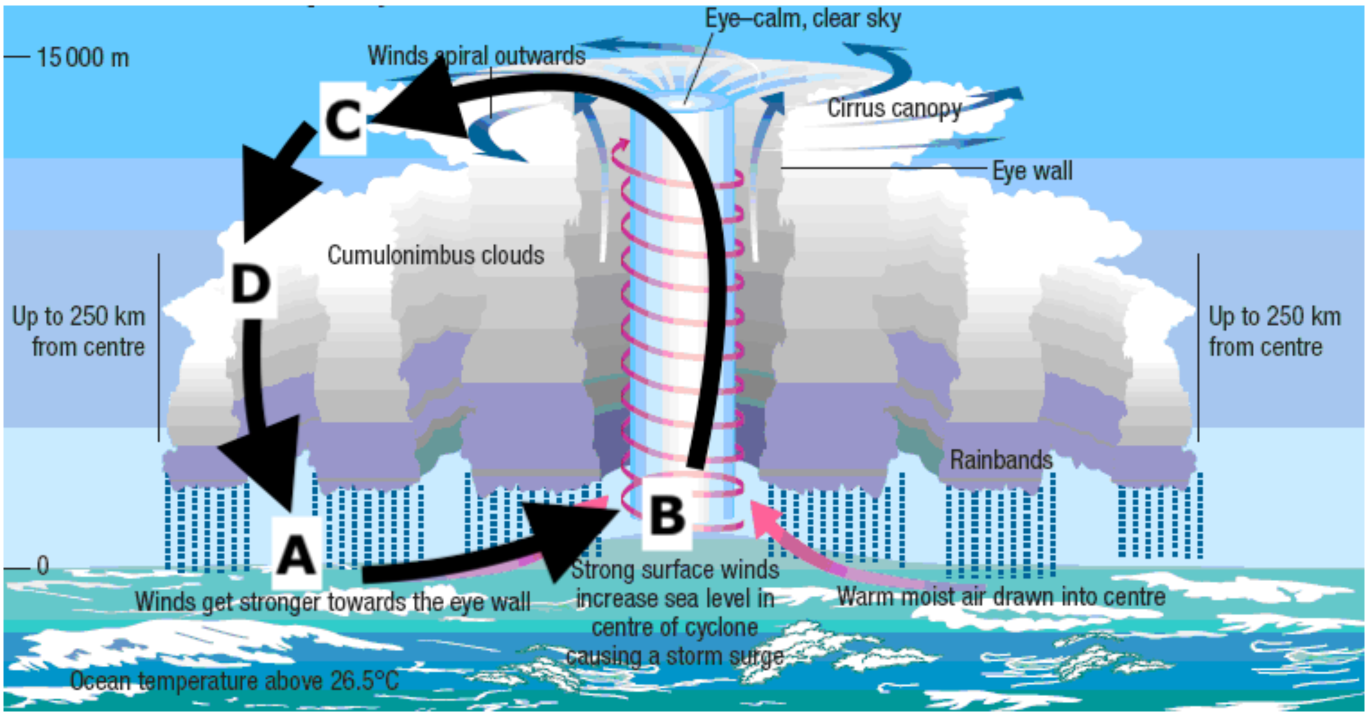


Figure 1: Schematic of the cyclone.

15. What is the effective mechanical power $\mathcal{P}'$ generated by the cyclone? We will introduce $\mathcal{D}$ the massive flow rate.
16. In which form is converted this energy? Given that $\mathcal{P}' \sim 10^{13}$ W, give an estimate of $\mathcal{D}$.

### Correction

1. We should consider the enthalpy $H = U + PV$. The first law of thermodynamics write

$$\Delta H + \Delta E_p + \Delta E_k = W' + Q \tag{1}$$

where $Q$ is the heat, $E_p$ the potential energy and $E_k$ the kinetic energy.

2. We have

$$\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}P \tag{2}$$

and

$$\mathrm{d}S = \frac{\delta Q}{T}. \tag{3}$$

3. The enthalpy evolution is

$$l_v \delta m_w = \Delta H_{AB} = T_c \Delta S_{AB} + \int V\mathrm{d}P = Q_{AB} + \frac{m_a}{M_a} R T_c \ln\left(\frac{P_B}{P_A}\right). \tag{4}$$

In particular,

$$\Delta S_{AB} = \frac{l_v}{T_c}\delta m_w - \frac{m_a}{M_a} R \ln\left(\frac{P_B}{P_A}\right). \tag{5}$$

Finally, the effective work is

$$W'_{AB} = \Delta H_{AB} - Q_{AB} + \frac{1}{2} m_a v_b^2 = \frac{m_a}{M_a} R T_c \ln\left(\frac{P_B}{P_A}\right) + \frac{1}{2} m_a v_b^2. \tag{6}$$

4. The cyclone provides an effective work $-W'_{AB}$ to the sea.
5. The ascension height is $\sim 10$ km, and the velocity $\sim 1\,\mathrm{m \cdot s^{-1}}$ thus $\tau_{as} \sim 10^3$ s. The diffusion time is typically $\tau_{diff} \sim \frac{L^2}{D_{diff}} \sim \frac{10^2}{10^{-5}} \sim 10^7$ s. Finally, the mechanical time is typically $\tau_{mech} \sim \frac{L}{c_s} \sim \frac{10}{10^2} \sim 10^{-1}$ s. The mechanical equilibrium is always valid but there is no diffusion is the time of the process.
6. The process is adiabatic and reversible so that $\Delta S_{BC} = Q_{BC} = 0$. The variation of enthalpy is

$$\Delta H_{BC} = -l_v \delta m_w + c_P (T_f - T_c) m_a. \tag{7}$$

Finally, the effective work is

$$0 = W'_{BC} = \Delta H_{BC} - \frac{1}{2} m_a v_b^2 + m_a g z_C = -l_v \delta m_w - c_P (T_c - T_f) m_a - \frac{1}{2} m_a v_b^2 + m_a g z_C. \tag{8}$$

7. We deduce
$$gz_C = l_v \frac{\delta m_w}{m_a} + c_P(T_c - T_f) + \frac{1}{2}v_b^2. \tag{9}$$
Since $l_v$ is large, the water allows $z_c$ to become larger.

8. We use the thermostatic of gas: $\nabla P = \rho \mathbf{g}$. Thus, $P_D = P_C - \rho g(z_D - z_C)$.

9. There is no change of enthalpy then
$$0 = \Delta H_{CD} = T_f \Delta S_{CD} + \frac{m_a}{M_a} RT_f \ln\left(\frac{P_D}{P_C}\right) \tag{10}$$
then
$$Q_{CD} = T_f \Delta S_{CD} = -\frac{m_a}{M_a} RT_f \ln\left(1 - \frac{\rho g}{P_C}(z_D - z_C)\right). \tag{11}$$
Finally, the effective work is
$$0 = W'_{CD} = m_a g(z_D - z_C) + \frac{m_a}{M_a} RT_f \ln\left(1 - \frac{\rho g}{P_C}(z_D - z_C)\right). \tag{12}$$

10. The process is adiabatic and reversible so that $\Delta S_{CD} = Q_{CD} = 0$. The variation of enthalpy is
$$\Delta H_{DA} = c_P(T_c - T_f)m_a. \tag{13}$$
Finally, the effective work is
$$0 = W'_{DA} = \Delta H_{DA} - m_a g z_D = c_P(T_c - T_f)m_a - m_a g z_D. \tag{14}$$

11. Thus,
$$gz_D = c_P(T_c - T_f) \tag{15}$$
and
$$g(z_D - z_C) = c_P(T_c - T_f) - l_v \frac{\delta m_w}{m_a} - c_P(T_c - T_f) - \frac{1}{2}v_b^2 = -l_v \frac{\delta m_w}{m_a} - \frac{1}{2}v_b^2 \tag{16}$$

12. The conservation of entropy writes
$$\frac{l_v}{T_c}\delta m_w - \frac{m_a}{M_a} R \ln\left(\frac{P_B}{P_A}\right) - \frac{m_a}{M_a} R \ln\left(1 - \frac{\rho g}{P_C}(z_D - z_C)\right) = 0. \tag{17}$$
For the thermostatic of the atmosphere, the typical length is 100 km which is large compared with the lengths considered here. Thus,
$$\ln\left(1 - \frac{\rho g}{P_C}(z_D - z_C)\right) \approx \frac{\rho g}{P_C}(z_D - z_C) = \frac{\rho}{P_C}\left(l_v \frac{\delta m_w}{m_a} + \frac{1}{2}v_b^2\right). \tag{18}$$
Using that $\frac{\rho R}{M_a P_C} = \frac{1}{T_f}$, we have
$$\frac{l_v}{T_c}\delta m_w - \frac{m_a}{M_a} R \ln\left(\frac{P_B}{P_A}\right) - \frac{m_a}{T_f}\left(l_v \frac{\delta m_w}{m_a} + \frac{1}{2}v_b^2\right) = 0 \Longrightarrow v_b^2 = \frac{2T_f R}{M_a} \ln\left(\frac{P_A}{P_B}\right) - 2\eta l_v \frac{\delta m_w}{m_a}. \tag{19}$$

13. The estimate is
$$v_b^2 \sim \frac{2 \cdot 200 \cdot 8}{30 \cdot 10^{-3}} \ln(10) - 2 \cdot (1 - 2/3) \cdot 5 \cdot 10^{-3} \cdot 2.5 \cdot 10^6 \sim 2 \cdot 10^5 - 8 \cdot 10^3 \sim 2 \cdot 10^5 \text{ usi} \tag{20}$$
then
$$v_b \sim 5 \cdot 10^2 \text{ m/s} \sim 2 \cdot 10^3 \text{ km/h}. \tag{21}$$

14. The effective force is $W' = -m_a \left(\frac{(T_c - T_f)R}{M_a} \ln\left(\frac{P_A}{P_B}\right) + \eta l_v \frac{\delta m_w}{m_a}\right)$ which is indeed negative. Thus,
$$\eta^* = \frac{\frac{(T_c - T_f)R}{M_a} \ln\left(\frac{P_A}{P_B}\right) + \eta l_v \frac{\delta m_w}{m_a}}{T_c l_v \frac{\delta m_w}{m_a} + \frac{T_c}{M_a} R \ln\left(\frac{P_A}{P_B}\right)} = \eta. \tag{22}$$
Our naive model is a Carnot machine.

15. The effective power is
$$\mathcal{P}' = -\frac{\mathcal{D}}{m_a} W' = \frac{\mathcal{D}}{M_a} RT_c \ln\left(\frac{P_A}{P_B}\right) - \frac{1}{2}\mathcal{D} v_b^2. \tag{23}$$

16. In mechanical energy. We estimate
$$\mathcal{D} = \frac{\mathcal{P}'}{\frac{(T_c - T_f)R}{M_a} \ln\left(\frac{P_A}{P_B}\right) + \eta l_v \frac{\delta m_w}{m_a}} \sim \frac{10^{13}}{10^5} \sim 10^8 \text{ m}^3/\text{s}. \tag{24}$$