TD 3: Thermoelectric Generator - Solutions

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In this exercise we consider a thermo-electric generator, *i.e.* we produce an electrical power from a gradient of temperature. Our model is described in figure 1. The hot thermal reservoir provides a heat flux $\dot{Q}_c$ while the cold thermal reservoir obtains a heat flux $\dot{Q}_f$. The electric circuit is a double P-N junction. Each semiconductor has its own resistivity $\rho_{P/N}$ and thermal conductivity $k_{P/N}$. In the first two sections of the exercise we only consider one of the semiconductor separately from the other. We denote $\mathcal{I}$ the electric current. The effective electric power $\mathcal{P}$ can be used to power an engine. We model it by a resistance R_L.

1. What is the most common way to produce electricity from a hot thermal reservoir?

Joule effect. We consider Joule effect to study $\dot{Q}(x)$ inside one of the semiconductors.

2. What is the physical origin of Joule effect?
3. Apply Joule effect to provide an expression for $\dot{Q}(x)$.

Fourier's law. This heat flux distribution must match with Fourier's law.

4. What is the physical origin of Fourier's law? Give its usual form.
5. Deduce $\dot{Q}(0)$ and $\dot{Q}(L)$.
6. Give the distribution of temperature. Comment.

Peltier effect. We now consider the P-N junctions.

7. Remind the Peltier effect.
8. Denoting $\alpha = \alpha_P - \alpha_N$, give the expression of $\dot{Q}_c$ and $\dot{Q}_f$.

Power and efficiency. We now consider the global system and denote $\rho = \rho_N + \rho_P$ and $k = k_N + k_P$.

9. Relate the electric current $\mathcal{I}$ with the resistance R_L and give the electrical power $\mathcal{P}$.
10. Give the expression of the efficiency $\eta = \frac{\mathcal{P}}{\dot{Q}_c}$. In which limit do we find Carnot efficiency η_C? Is it physically plausible?
11. What is the free parameter in the system? What is the maximal value of $\mathcal{I}$ denoted $\mathcal{I}_m$.
12. Show that
$$\eta = \eta_C \frac{j(1-j)}{j - \frac{1}{2}\eta_C j^2 + A} \tag{1}$$
where $j = \frac{\mathcal{I}}{\mathcal{I}_m}$, η_C is the Carnot efficiency and $A = \frac{KR}{\alpha^2 T_C}$.

Power-efficiency diagram. We now study the efficiency and power as a function of j.

13. Calculate the maximal power $\mathcal{P}_m$ and the associated current. Deduce that
$$\frac{\mathcal{P}}{\mathcal{P}_m} = 4j(1-j). \tag{2}$$
14. Calculate the current j^* at maximal efficiency. We will assume $A \gg 1$.
15. Draw the diagram $\left(\frac{\mathcal{P}}{\mathcal{P}_m}, \eta\right)$. Comment.

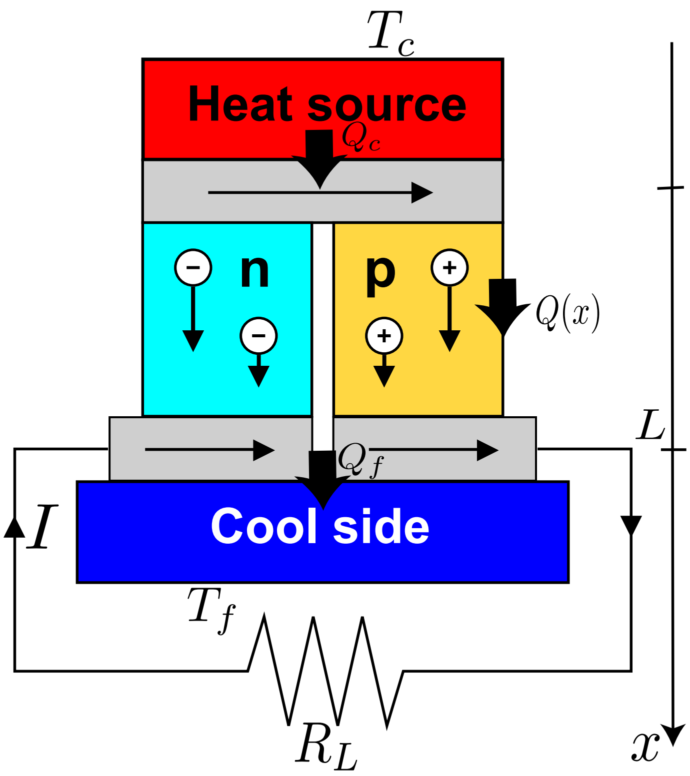

Figure 1: Schematic of the thermoelectric generator.

Correction

1. The most common way is to vaporise water and to use a turbine with a dynamo generator.
2. The movement of electrons in the solid comes with collisions between the electrons and the atoms, generating heat.
3. We look between and x and $x + \mathrm{d}x$. Joule effect produces a heat $\mathrm{d}\dot{Q} = \frac{\rho \mathcal{I}^2}{S}\mathrm{d}x$ where S is the section of the semiconductor. We deduce $\dot{Q}(x) = \frac{\rho \mathcal{I}^2}{S}x + \dot{Q}(0) = R\mathcal{I}^2\frac{x}{L} + \dot{Q}(0)$ where $R = \frac{\rho L}{S}$ is the total resistance.
4. Fourier's law comes from diffusion. Hotter molecules will collide with neighbouring molecules and the temperature will equilibrate. Its usual form is $\mathbf{j}_q = -k\nabla T$.
5. Thus, $\dot{Q}(x) = -kS\partial_x T$ and then $kST(x) = -\frac{R\mathcal{I}^2}{2}\frac{x^2}{L} - \dot{Q}(0)x + C$. Using the fixed temperatures at $x = 0$ and $x = L$ we deduce $C = kST_c$ and $\dot{Q}(0) = K(T_c - T_f) - \frac{R\mathcal{I}^2}{2}$, where $K = \frac{kS}{L}$. Finally, $\dot{Q}(L) = K(T_c - T_f) + \frac{R\mathcal{I}^2}{2}$
6. We deduce $T(x) = -\frac{R\mathcal{I}^2}{2K}\frac{x^2}{L^2} - (T_c - T_f - \frac{R\mathcal{I}^2}{2K})\frac{x}{L} + T_c = \frac{R\mathcal{I}^2}{2K}\frac{x}{L}(1 - \frac{x}{L}) + T_c + (T_f - T_c)\frac{x}{L}$. We see that the temperature is not decreasing, which is surprising. The maximum temperatur is obtained for
$$0 = \frac{R\mathcal{I}^2}{2K}(1 - 2\frac{x}{L}) + T_f - T_c \Longrightarrow \frac{x}{L} = \frac{1}{2} - \frac{K}{R\mathcal{I}^2}(T_c - T_f). \quad (3)$$
7. The electric current transports heat $\mathbf{j}_q = \alpha T\mathbf{j}$. At a junction, the same current cannot transport the same amount of heat in which side of the junction. For a current going from A to B, the difference of heat $\dot{Q}^P_{AB} = (\alpha_A - \alpha_B)T\mathcal{I}$ is lost by the system.
8. We have $\dot{Q}_c = \dot{Q}(0) - \dot{Q}^P_{NP} = K(T_c - T_f) - \frac{R\mathcal{I}^2}{2} + \alpha T_c\mathcal{I}$ and $\dot{Q}_f = \dot{Q}(L) + \dot{Q}^P_{PN} = K(T_c - T_f) + \frac{R\mathcal{I}^2}{2} + \alpha T_f\mathcal{I}$.
9. From the first principle we have $\mathcal{P} = \dot{Q}_c - \dot{Q}_f = \alpha(T_c - T_f)\mathcal{I} - R\mathcal{I}^2$. On the other side, $\mathcal{P} = R_L\mathcal{I}^2$. This provides $\mathcal{I} = \alpha\frac{T_c - T_f}{R + R_L}$.
10. We calculate
$$\eta = \frac{\mathcal{P}}{\dot{Q}_c} = \frac{\alpha(T_c - T_f)\mathcal{I} - R\mathcal{I}^2}{K(T_c - T_f) - \frac{R\mathcal{I}^2}{2} + \alpha T_c\mathcal{I}}. \quad (4)$$
We find Carnot efficiency if $R \to 0$ and $K \to 0$, *i.e.* a material with no thermal conductivity and infinite electric conductivity. This unphysical. In particular, this contradicts the Wiedman-Franz law which states $\frac{k\rho}{T} \approx \frac{\pi^3}{3}\left(\frac{k_B}{e}\right)^2$.
11. The free parameter is the system we power, *i.e.* R_L. This free parameter control $\mathcal{I}$ which ranges between 0 and $\mathcal{I}_m = \alpha\frac{T_c - T_f}{R}$.

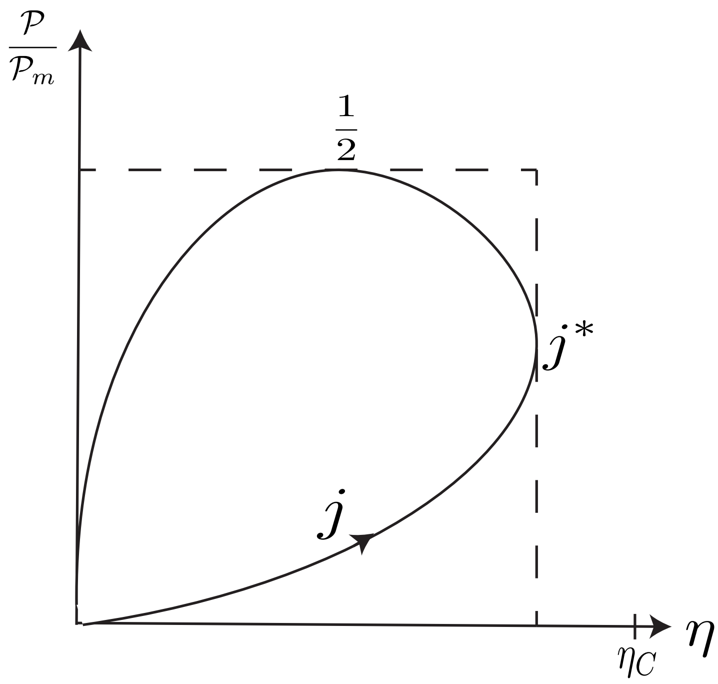

Figure 2: Diagram power-efficiency.

12. We calculate

$$\eta = \frac{\mathcal{P}}{\dot{Q}_c} = \frac{\alpha(T_c - T_f)\mathcal{I} - R\mathcal{I}^2}{K(T_c - T_f) - \frac{R\mathcal{I}^2}{2} + \alpha T_c \mathcal{I}} = \eta_C \frac{j(1-j)}{j - \frac{1}{2}\eta_C j^2 + \frac{KR}{\alpha^2 T_C}}. \quad (5)$$

13. We have $\mathcal{P} = \alpha(T_c - T_f)\mathcal{I} - R\mathcal{I}^2$. Its maximum is obtained for

$$0 = \alpha(T_c - T_f) - 2R\mathcal{I} \Longrightarrow \mathcal{I} = \alpha\frac{T_c - T_f}{2R} \Longrightarrow j = \frac{1}{2}. \quad (6)$$

Thus, the maximal power is $\mathcal{P}_m = \frac{\alpha^2(T_c - T_f)^2}{4R}$. We deduce easily

$$\frac{\mathcal{P}}{\mathcal{P}_m} = 4j(1-j). \quad (7)$$

14. η is maximum for

$$(1-2j)(j - \frac{1}{2}\eta_C j^2 + A) = j(1-j)(1-\eta_C j) \Longrightarrow j^2(1-\eta_C/2) + 2Aj - A = 0. \quad (8)$$

Thus, $j^* = \frac{\sqrt{A^2 + A(1-\eta_C/2)} - A}{1-\eta_C/2} = \frac{A}{1-\eta_C/2}\left(\sqrt{1 + \frac{1-\eta_C/2}{A}} - 1\right) \approx \frac{1}{2} - \frac{1-\eta_C/2}{4A}$.

15. We notice that both the power and efficiency vanish when $j = 0$ or $j = 1$, which is different from the Curzon-Alhborne diagram. We see an hysteresis. At fixed power there are two possible efficiencies depending on the choice of the current. Thus, the interesting part of the diagram is obtained for

$$j^* < j < \frac{1}{2}. \quad (9)$$