TD 4: Entropy of a Ray of Light - Solutions

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1 Maximal Work from a Gas of Photons: Carnot Cycle. In this exercise we consider a hot source of volume V at temperatureT_1 emitting light to a cold source at temperature T_2. The hot source is a black body then Stephan's law applies: the emitted power is $\mathcal{P} = \frac{cU}{4V} = \sigma T^4$ where U is the internal energy, c the speed of light and σ is Stephan's constant. Moreover, the pressure can be expressed as $P = \frac{U}{3V}$. We are interested in the maximal work which can be extracted and therefore we assume that we have a Carnot machine to do the transfer.

1. Calculate the heat capacity C of the gas of photons. Is it a heat reservoir? We then denote $T(t)$ the temperature of the hot source, T_1 being its initial temperature. The cold source is assumed large enough to be a thermal reservoir.
2. We denote $\dot{Q}_1$ (resp. $\dot{Q}_2$) the heat provided by the hot source (resp. the cold source) at temperature T (resp. T_2) to the system and $\dot{W}$ the work provided to the system. Give the expression of $\dot{W}$ as a function of C, T and T_2.
3. Calculate the total work W which can be extracted from the gas of photons as a function of $U(T_1)$, T_1 and T_2.
4. Calculate the associated efficiency.
5. The surface temperature of the Sun is $T_1 \sim 4000$ K while for the Earth $T_2 \sim 300$ K. Give an order of magnitude of the maximum efficiency for extracted work from from sunlight.

Correction

1. We have $U = \frac{4\sigma V}{c} T^4$. Then, $C = \left(\frac{\partial U}{\partial T}\right)_V = \frac{16\sigma V}{c} T^3$. The heat capacity is finite therefore this is not a thermal reservoir, the temperature changes.
2. The first principle gives $\dot{W} + \dot{Q}_1 + \dot{Q}_2 = 0$. The second principle gives $\frac{\dot{Q}_1}{T} + \frac{\dot{Q}_2}{T_2} = 0$. Thus
$$\frac{\dot{Q}_1}{T} = -\frac{\dot{Q}_2}{T_2} = \frac{\dot{Q}_1}{T_2} + \frac{\dot{W}}{T_2} \Longrightarrow \dot{W} = -\left(1 - \frac{T_2}{T}\right)\dot{Q}_1 = C(T)\left(1 - \frac{T_2}{T}\right)\dot{T}. \tag{1}$$
3. The total work is
$$W = \int \dot{W} \mathrm{d}t = \int \frac{16\sigma V}{c} T^3 \left(1 - \frac{T_2}{T}\right)\dot{T}\mathrm{d}t = \frac{16\sigma V}{c}\int_{T_2}^{T_1} \left(T^3 - T_2 T^2\right)\mathrm{d}T \tag{2}$$
$$= \frac{16\sigma V}{c}\left(\frac{T_1^4 - T_2^4}{4} - T_2\frac{T_1^3 - T_2^3}{3}\right) = U(T_1)\left(1 + \frac{1}{3}\left(\frac{T_2}{T_1}\right)^4 - \frac{4}{3}\frac{T_2}{T_1}\right). \tag{3}$$
4. The efficiency writes
$$\eta = \frac{W}{Q_1} = \frac{W}{U(T_1) - U(T_2)} = \frac{U(T_1)}{U(T_1) - U(T_2)}\left(1 + \frac{1}{3}\left(\frac{T_2}{T_1}\right)^4 - \frac{4}{3}\frac{T_2}{T_1}\right) \tag{4}$$
$$= \frac{1 + \frac{1}{3}\left(\frac{T_2}{T_1}\right)^4 - \frac{4}{3}\frac{T_2}{T_1}}{1 - \left(\frac{T_2}{T_1}\right)^4} = 1 - \frac{4}{3}\frac{T_2}{T_1}\frac{1 - \left(\frac{T_2}{T_1}\right)^3}{1 - \left(\frac{T_2}{T_1}\right)^4} \tag{5}$$
5. For the Sun and the Earth, we find $\eta \sim 90\%$ which is not so bad. But here the transfer is reversible.

2 Maximal Work from a Gas of Photons: Thermodynamic Potential. We consider a system of physical properties T, P, V, S, U immersed in a reservoir of volume V_0, pressure P_0, temperature T_0 and entropy S_0. We denote $A = U - T_0 S + P_0 V$ the thermodynamic potential.

1. Prove that A is the thermodynamic potential for this problem.

   We go back to the gas of photons and use the same notations than in the previous exercise. We assume the system 2 to be both a gas of photons and a reservoir.

2. Express the entropy S as a function of V and T.

3. Deduce the maximal work which can be extracted from the gas of photons. Notice that this result does not depend on the change of volume ΔV.

Correction

1. We look at the internal energy U during the process. We denote W the effective work provided by the reservoir. The forces of pressure generate an additional work $P_0 \Delta V_0$, where we have used that P_0 was preserved. Similarly, the reservoir gives a heat $T_0 \Delta S_0$, taking advantage of the preservation of T_0. Everything together, the change of energy for the system is

$$\Delta U = W + P_0 \Delta V_0 - T_0 \Delta S_0. \tag{6}$$

   The total volume does not change so that $\Delta V_0 = -\Delta V$. The second principle of thermodynamics states that $\Delta S_0 + \Delta S \geq 0$. Thus,

$$\Delta U \geq W - P_0 \Delta V - T_0 \Delta S \Longrightarrow W \leq \Delta A. \tag{7}$$

   We conclude that A is the right thermodynamic potential.

2. We have $P = \frac{U}{3V} = \frac{4\sigma}{3c} T^4$. The free energy is $F = U - TS$, with variations $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V$ *i.e.* $\left(\frac{\partial F}{\partial V}\right)_T = -P = -\frac{4\sigma}{3c} T^4$. We then deduce $F = -\frac{4\sigma}{3c} V T^4$. In the same time, $S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{16\sigma}{3c} V T^3$.

3. We use the thermodynamic potential from (T_1, V) to $(T_2, 0)$:

$$\begin{aligned} W_{max} &= -\Delta A = -\Delta U + T_2 \Delta S - P_2 \Delta V = \frac{4\sigma}{c} V T_1^4 - T_2 \frac{16\sigma}{3c} V T_1^3 + \frac{4\sigma}{3c} T_2^4 V \qquad (8) \\ &= U(T_1) \left(1 + \frac{1}{3}\left(\frac{T_2}{T_1}\right)^4 - \frac{4}{3}\frac{T_2}{T_1}\right). \qquad (9) \end{aligned}$$

   One can notice that this does not depends on ΔV. Indeed, if we used a final volume V_f, it would have added the terms:

$$-\frac{4\sigma}{c} V_f T_2^4 + T_2 \times \frac{16\sigma}{3c} V_f T_2^3 - \frac{4\sigma}{3c} T_2^4 V_f = 0. \tag{10}$$

3 Entropy of Radiation. In this exercise we consider a ray of light. We work in the micro-canonical ensemble but we do not assume thermodynamic equilibrium. We define $\mathbb{P}(N_1, N_2, \ldots)$ the probability to find N_1 photons in the state 1, N_2 photons in the state 2. We make 2 assumptions on the form of $\mathbb{P}$. We assume the probability distribution to be of the form $\mathbb{P}(N_1, N_2, \ldots) = p_1(N_1) p_2(N_2) \ldots$. Moreover we assume that the probability to find an electron at state j does not depend on the numbers of photons at the state j.

1. What is the physical meaning o theses 2 assumptions.

2. Show that the entropy is

$$S = -k_B \sum_j \sum_{N_j=0}^{\infty} p_j(N_j) \ln(p_j(N_j)). \tag{11}$$

3. Justify that p_j follows a geometric law and give its expression.

4. We denote n_j the average number of photons in the state j. Express p_j as a function of n_j.

5. Deduce an expression of the entropy as a function of the (n_j).

   We consider a surface $\mathrm{d}A$ from which the ray of light is emitted in the direction $\mathrm{d}\Omega$ which make an angle θ with the normal to the surface. We remind that the number of states of frequencies between ν and $\nu + \mathrm{d}\nu$ can be estimated by $2V \frac{\nu^2}{c^3} \mathrm{d}\nu \mathrm{d}\Omega$.

6. Express s_ν as a function of n_ν and ν such that the entropy transported by the ray is

$$S = V \int s_\nu \mathrm{d}\nu \mathrm{d}\Omega. \tag{12}$$

7. Similarly, express u_ν as a function of n_ν and ν such that the internal energy transported by the ray is

$$U = V \int u_\nu \mathrm{d}\nu \mathrm{d}\Omega. \tag{13}$$

8. We consider the ray at frequency centered at ν_0 with width $\Delta\nu$ and directed to the normal to the surface with width $\Delta\Omega$. Express the flux of energy and entropy per unit surface.

9. Define an effective temperature of the ray. At what condition the flow of energy is non-vanishing for $\Delta\Omega\Delta\nu \to 0$. What is the flow of entropy in this case. Comment.

10. We provide an energy flux $\dot{W}$ and a heat flux $\dot{Q}$ to a light which emits a power $\dot{E}$ and an entropy $\dot{S}$. What is the maximal efficiency when the entropy flux is vanishing? When the system is reversible?

Correction

1. We have assumed no interactions between the photons.

2. The entropy is

$$\begin{aligned} S &= -k_B \sum_{\{N_j\}} \mathbb{P}(N_1, N_2, \dots)\ln(\mathbb{P}(N_1, N_2, \dots)) = -k_B \sum_{\{N\}} \prod_k p_k(N_k) \sum_j \ln(p_j(N_j)) && (14)\\ &= -k_B \sum_j \left(\prod_{k\neq j} \sum_{N_k} p_k(N_k) \right) \sum_{N_j} p_j(N_j)\ln(p_j(N_j)) = -k_B \sum_j \sum_{N_j=0}^{\infty} p_j(N_j)\ln(p_j(N_j)). && (15) \end{aligned}$$

3. We have assumed that the probability to find an electron at state j does not depend on the numbers of photons at the state j. This is the definition of a no-memory law. Thus, p_j follows a geometric law: $p(k) = (1-q)q^k$.

4. It is standard to prove that the mean value of a geometric law is $n = \sum_{k=0}^{\infty} k(1-q)q^k = \frac{q}{1-q}$. Thus, $q = \frac{n}{1+n}$. We deduce $p_j(k) = \frac{n_j^k}{(1+n_j)^{k+1}}$.

5. We express the entropy:

$$\begin{aligned} S &= -k_B \sum_j \sum_{N_j=0}^{\infty} \frac{n_j^{N_j}}{(1+n_j)^{N_j+1}} \ln\left(\frac{n_j^{N_j}}{(1+n_j)^{N_j+1}} \right) && (16)\\ &= -k_B \sum_j \left(\ln\left(\frac{n_j}{1+n_j}\right) \sum_{N_j=0}^{\infty} \frac{N_j n_j^{N_j}}{(1+n_j)^{N_j+1}} - \ln(1+n_j) \sum_{N_j=0}^{\infty} \frac{n_j^{N_j}}{(1+n_j)^{N_j+1}} \right) && (17)\\ &= -k_B \sum_j \left(\ln\left(\frac{n_j}{1+n_j}\right) \sum_{N_j=0}^{\infty} N_j p_j(N_j) - \ln(1+n_j) \sum_{N_j=0}^{\infty} p_j(N_j) \right) && (18)\\ &= -k_B \sum_j \left(\ln\left(\frac{n_j}{1+n_j}\right) n_j - \ln(1+n_j) \right) && (19)\\ &= k_B \sum_j \left((1+n_j)\ln(1+n_j) - n_j \ln\left(n_j\right) \right). && (20) \end{aligned}$$

6. We go to the continuum limit:

$$S = k_B \int \left((1+n_\nu)\ln(1+n_\nu) - n_\nu \ln\left(n_\nu\right) \right) 2V\frac{\nu^2}{c^3} \mathrm{d}\nu \mathrm{d}\Omega. \tag{21}$$

Thus, $s_\nu = 2k_B \frac{\nu^2}{c^3} \left((1+n_\nu)\ln(1+n_\nu) - n_\nu \ln\left(n_\nu\right) \right)$.

7. Similarly the entropy of a photon being $h\nu$ we have

$$U = \int h\nu n_\nu \times 2V\frac{\nu^2}{c^3} \mathrm{d}\nu \mathrm{d}\Omega. \tag{22}$$

Thus, $u\nu = 2k_B h \frac{\nu^3}{c^3} n_\nu$.

8. The volume is $V = c\Delta t \mathrm{d}A$. The flux of energy then becomes $\phi_U = \frac{2h\nu_0^3}{c^2} n_{\nu_0} \Delta\nu\Delta\Omega$ while the flux of entropy becomes $\phi_S = \frac{2k_B\nu_0^2}{c^2} \left((1+n_{\nu_0})\ln(1+n_{\nu_0}) - n_{\nu_0} \ln\left(n_{\nu_0}\right) \right) \Delta\nu\Delta\Omega$.

9. An effective temperature is

$$T_F = \frac{\phi_U}{\phi_S} = \frac{h\nu_0}{k_B}\frac{n_{\nu_0}}{(1+n_{\nu_0})\ln(1+n_{\nu_0}) - n_{\nu_0}\ln\left(n_{\nu_0}\right)}. \tag{23}$$

When $\Delta\Omega\Delta\nu \to 0$, the flux of energy is non-vanishing for $n_{\nu_0} \to \infty$. In that case, $T_F \to \infty$ *i.e.* $\phi_S \to 0$. When the ray o light is whether unidirectional or monochromatic, it does not transport entropy.

10. We provide a work $\dot{W}$ and a heat $\dot{Q}$ to the light which emits a power $\dot{E}$ and an entropy $\dot{S}$ through light which has an effective temperature T_F. The first principle states that $\dot{W} - \dot{E} + \dot{Q} = 0$ while the second principle states that $0 = \frac{\dot{Q}}{T} - \dot{S} + \dot{S}_c$ where $\dot{S}_c \geq 0$. When the energy flux vanishes $\dot{S} = 0$, then $\dot{Q} = 0$ and $\dot{W} = \dot{E}$ *i.e.* $\eta = \frac{\dot{E}}{\dot{W}} = 1$. We just transfer the work, there is no heat. Now we just have that the system is reversible *i.e.* $\dot{S}_c = 0$. We then deduce $\dot{W} = \dot{E} - T\dot{S} = \left(1 - \frac{T}{T_F}\right)\dot{E}$. Thus, the efficiency is $\eta = \frac{T_F}{T_F - T}$. When choosing $T_F \to T$ the efficiency diverges. In that case most of the energy comes from $\dot{Q} = \frac{T}{T_F}\dot{E}$.