TD 7: Large Deviations of Radioactive Decay

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We want to study the radioactive decay of some material. We denote $(x_n)_{i=1}^N$ the several particles, whose value is 1 is the particle remains excited and 0 if it has already disintegrated. Thus, $x_i \in \{0,1\}$ with initial condition $x_i(t=0) = 1$ and a decay rate λ, *i.e.* its probability of disintegration at time t is $\lambda e^{-\lambda t}$. We assume the particles to be independent and denote

$$X_N(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \tag{1}$$

the density of radioactive particles.

1. What is the probability distribution of x_i.

Thermodynamic Limit. We consider the thermodynamic limit $N \to +\infty$.

2. Deduce the limit of X_N using the large number law.
3. What equation is fulfilled by this limit? Recall its physical meaning.

Dynamic Formalism for Large Deviations. We now want to study the large deviations of X_N, which depends on time t. We then want the dynamic version of X_N for $0 \leq t \leq T$. We start by considering M discrete time steps of length $\mathrm{d}t = \frac{T}{M}$. We then denote $X_N^j = X_N(j\mathrm{d}t)$. At the end, we are interested in the limit $M \to +\infty$. For teh moment we do not specify the law of x_i so that we remain general.

4. We denote $\mathbb{P}_x$ the probability given than the initial condition is $X_N = x$. Express
$$\mathbb{P}_{x_0}(X_N^1 = x_1, \dots X_N^M = x_M) \tag{2}$$
using only the random variable X_N^1.
5. Deduce that for any p,
$$\mathbb{P}_{x_0}(X_N^1 = x_1, \dots X_N^M = x_M) \leq \exp\left(-N\mathrm{d}t\sum_{j=0}^{M-1}\left(p\frac{x_{j+1}-x_j}{\mathrm{d}t} - \mathcal{H}_M(x_j,p)\right)\right) \tag{3}$$
where we have defined
$$\mathcal{H}_M(x,p) = \frac{1}{N\mathrm{d}t}\ln\left(\mathbb{E}_x\left(e^{Np(X_N^1-x)}\right)\right). \tag{4}$$
6. Going to the limit $M \to +\infty$, $\mathcal{H}_M(x,p) \to \mathcal{H}(x,p)$, the effective Hamiltonian. We consider a path $x(t)$ where we sample $x_j = x(j\mathrm{d}t)$. Deduce an upper bond of $\mathbb{P}_{x_0}(X_N(t) = x(t) \text{ for } 0 \leq t \leq T)$.
7. Optimizing $p \in \mathbb{R}$, we perform a Legendre transform and find the Lagrangian
$$\mathcal{L}(x,\dot{x}) = \sup_{p\in\mathbb{R}}\left(p\dot{x} - \mathcal{H}(x,p)\right). \tag{5}$$
The inequality then becomes almost an equality (actually a log-equivalence). Define the action of large deviation and state the result of this section.

Application to the Radioactive Decay. We now come back to the initial problem.

8. Calculate the large deviation Hamiltonian.
9. Deduce that the Lagrangian is
$$\mathcal{L}(x,\dot{x}) = \lambda x + \dot{x}\left(1 - \ln\left(-\frac{\dot{x}}{\lambda x}\right)\right) \tag{6}$$
where we have assumed $\dot{x} < 0$ which is true physically.
10. Deduce the most probable trajectory? Comment.
11. Let us denote $\dot{x}(t=0) = -\lambda + v$ and let us remind that $x(t=0) = 1$. Estimate $\mathcal{L}(x,\dot{x})$ for small v. Deduce the right initial condition and then the equation of the most probable trajectory.

Estimate of a Deviation. We consider a time T. We have proven that $\bar{x}(T) = \mathbb{E}(X_N(T)) = e^{-\lambda T}$. We are interested in a deviation to a and we want to estimate the probability that $X_N(T) = a$.

12. Going back to the time discretization, express the upper bound of the probability $\mathbb{P}(X_N(T) = a)$ using the discrete Hamiltonian.

13. When going to the limit $M \to +\infty$, there is an infinite number of integrals: this is a path integral. The probability then becomes:

$$\mathbb{P}(X_N(T) = a) = \int_{x(0)=1}^{x(T)=a} \mathcal{D}x \, e^{-N\mathcal{S}(x)}. \tag{7}$$

How to deduce the Gaussian approximation of this probability?

——— *Only when you have finished all the exercises* ———

The Wikipedia Moment. JAMES CLERK MAXWELL (1831-1879).

James Clerk Maxwell was born on 13 June 1831 in Edinburgh, to John Clerk Maxwell of Middlebie, an advocate. His father was a man of comfortable means of the Clerk family of Penicuik, holders of the baronetcy of Clerk of Penicuik. He had been born "John Clerk", adding Maxwell to his own after he inherited the Middlebie estate, a Maxwell property in Dumfriesshire.

Recognising the boy's potential, Maxwell's mother Frances took responsibility for his early education, which in the Victorian era was largely the job of the woman of the house. James' died in December 1839 when he was eight years old. His education was then overseen by his father and his father's sister-in-law Jane, both of whom played pivotal roles in his life. Maxwell was sent to the prestigious Edinburgh Academy. Maxwell's interests ranged far beyond the school syllabus and he did not pay particular attention to examination performance. He wrote his first scientific paper at the age of 14. The work, of 1846, "On the description of oval curves and those having a plurality of foci" was presented to the Royal Society of Edinburgh by James Forbes, a professor of natural philosophy at the University of Edinburgh, because Maxwell was deemed too young to present the work himself.

Maxwell left the Academy in 1847 at age 16 and began attending classes at the University of Edinburgh. He did not find his classes demanding, and was therefore able to immerse himself in private study during free time at the university and particularly when back home at Glenlair. Through this practice he discovered photoelasticity, which is a means of determining the stress distribution within physical structures, and published several other articles.

In October 1850, already an accomplished mathematician, Maxwell left Scotland for the University of Cambridge where he attended Trinity college. At Trinity he was elected to the elite secret society known as the Cambridge Apostles. In 1854, Maxwell graduated with a degree in mathematics. In 1856, he accepted the professorship at Aberdeen. The 25-year-old Maxwell was a good 15 years younger than any other professor. He engaged himself with his new responsibilities as head of a department, devising the syllabus and preparing lectures. He committed himself to lecturing 15 hours a week.

In 1857 Maxwell befriended the Reverend Daniel Dewar, who was then the Principal of Marischal. Through him Maxwell met Dewar's daughter, Katherine Mary Dewar. They were engaged in February 1858 and married in Aberdeen on 2 June 1858. On the marriage record, Maxwell is listed as Professor of Natural Philosophy in Marischal College, Aberdeen. Katherine was seven years Maxwell's senior.

In 1859, Maxwell read Rudolf Clausius's work which inspired him a lot. From basic reasoning, he calculated the fraction of molecules whose velocity is between v and $v + \mathrm{d}v$. His results are published in 1860 in the article "Illustrations of the Dynamical Theory of Gases".

In 1860 Marischal College merged with the neighbouring King's College to form the University of Aberdeen. There was no room for two professors of Natural Philosophy, so Maxwell, despite his scientific reputation, found himself laid off. Maxwell was granted the Chair of Natural Philosophy at King's College, London.

Maxwell's time at King's was probably the most productive of his career. This period of his life would see him display the world's first light-fast colour photograph, further develop his ideas on the viscosity of gases, and propose a system of defining physical quantities—now known as dimensional analysis. This time is especially noteworthy for the advances Maxwell made in the fields of electricity and magnetism, leading to what is now known as Maxwell's equations.

In 1865 Maxwell resigned the chair at King's College, London, and returned to Glenlair with Katherine. In his paper 'On governors' (1868) he mathematically described the behaviour of governors, devices that control the speed of steam engines, thereby establishing the theoretical basis of control engineering. In his paper "On reciprocal figures, frames and diagrams of forces" (1870) he discussed the rigidity of various designs of lattice. He wrote the textbook Theory of Heat (1871) and the treatise Matter and Motion (1876). Maxwell was also the first to make explicit use of dimensional analysis, in 1871.

In 1871 he returned to Cambridge as Professor of Physics. Maxwell died in Cambridge of abdominal cancer on 5 November 1879 at the age of 48.