TD 8: Markov Processes - Solutions

Baptiste Coquinot & Antonio Costa

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A random process X_t can be viewed as a family of random numbers, indexed by the label t. For each time t, X_t may obey a different probability distribution $p(x,t)$. The values of the random process at different times t, t' may or may not depend on each other. The conditional probability $p(x_n, t_n | x_{n-1}, t_{n-1}, ...x_1, t_1)$ is defined as the probability of X_{t_n} taking the value x_n, given that X_{t_i} takes the value x_i for each $i \in \{1, ..., n-1\}$. If

$$p(x_n, t_n | x_{n-1}, t_{n-1}; ...; x_1, t_1) = p(x_n, t_n), \tag{1}$$

X_t is a *purely random process*, where the values of X_t at different times are independent, which cannot describe a physical continuous dependence on time. The second simplest case,

$$p(x_n, t_n | x_{n-1}, t_{n-1}; ...; x_1, t_1) = p(x_n, t_n | x_{n-1}, t_{n-1}), \tag{2}$$

defines a *Markov process*. One also calls $p(x, t | x', t')$ *transition probability*.

Basics of Markov Chains.

1. Show that for Markov process, the n-point joint probability density reduces to

$$p(x_n, t_n; ...; x_1, t_1) = p(x_n, t_n | x_{n-1}, t_{n-1}) p(x_{n-1}, t_{n-1} | x_{n-2}, t_{n-2}) ... p(x_2, t_2 | x_1, t_1) p(x_1, t_1). \tag{3}$$

2. Show further that this implies

$$p(x_3, t_3 | x_1, t_1) = \int p(x_3, t_3 | x_2, t_2) p(x_2, t_2 | x_1, t_1) \mathrm{d}x_2. \tag{4}$$

   This relation is known as Chapman-Kolmogorov equation.

3. *(Bonus)* For pure Brownian motion, the transition probability is:

$$p(x_2, t_2 | x_1, t_1) = \frac{1}{\sqrt{4\pi(t_2 - t_1)}} e^{\frac{-(x_2 - x_1)^2}{4(t_2 - t_1)}},$$

   meaning that they depend only on the difference in positions and times. Show that such transition probability satisfies the Chapman-Kolmogorov equation.

The Master Equation. Consider the transition probability from some state x'' at time t to another state x at time $t + \Delta t$ for Δt small,

$$p(x, t + \Delta t | x'', t) = (1 - a(x, t)\Delta t)\delta(x - x'') + W(x, x'', t)\Delta t + O(\Delta t^2). \tag{5}$$

Here the term involving $\delta(x - x'')$ is the probability to be at the same point after Δt, while $W(x, x'', t)$ (*the rate function*) is the probability to transition from x'' to x within the time interval Δt.

4. Determine $a(x, t)$ from the constraint of normalisation.

5. Use the Chapman-Kolmogorov equation to show that

$$\partial_t p(x, t | x', t') = \int \left[W(x, x'', t) p(x'', t | x', t') - W(x'', x, t) p(x, t | x', t') \right] \mathrm{d}x''. \tag{6}$$

   This is the so-called *continuous-time master equation*, which implies,

$$\partial_t p(x, t) = \int \left[W(x, x', t) p(x', t) - W(x', x, t) p(x, t) \right] \mathrm{d}x'. \tag{7}$$

The Fokker-Planck Equation. We now want to perform an expansion to find a partial differential equation describing our process.

6. Write $W(x,x',t) = w(x',r,t)$ with $r = x - x'$. Show that the Master equation implies

$$\partial_t p(x,t) = \int [w(x-r,r,t)p(x-r,t) - w(x,-r,t)p(x,t)]\,\mathrm{d}r. \tag{8}$$

Expand the first argument of $w(x-r,r,t)p(x-r,t)$ around x (*Kramers-Moyale expansion*) to show that

$$\partial_t p(x,t) = \sum_{n=1}^{\infty} (-\partial_x)^n \left[D_n(x,t)p(x,t)\right], \tag{9}$$

where $D_n = \frac{1}{n!}\int w(x,r,t)r^n \mathrm{d}r$. This series may terminate at order 2, in which case we obtain the Fokker-Planck equation:

$$\partial_t p(x,t) = -\partial_x[D_1(x,t)p(x,t)] + \partial_x^2[D_2(x,t)p(x,t)]. \tag{10}$$

7. Show that the Fokker-Planck equation can be written as a conservation law $\partial_t p = \partial_x J$, write down J.

8. Assume $x \in \mathbb{R}$ and $p(x,t) \overset{x\to\pm\infty}{\to} 0$ sufficiently fast. What equation does the mean $\langle x\rangle$ obey?

9. Given two solutions $p_1(x,t), p_2(x,t)$ of the Fokker-Planck equation starting from different initial conditions, consider $H(t) = \int p_1 \ln(p_1/p_2)\mathrm{d}x$, that we assume well defined. Show that $H(t) \geq 0$ and that $\frac{\mathrm{d}}{\mathrm{d}t}H(t) \leq 0$. What does this tell us about the long-time behaviour of the solutions? Discuss.

Correction

1. First simplify the notation by letting $x_i, t_i \equiv "i"$. Use the definition of conditional probabilities, then the Markov property, iterate:

$$p(n;n-1;...;1) \overset{def}{=} \underbrace{p(n|n-1;...;1)}_{=p(n|n-1)} \underbrace{p(n-1;...;1)}_{=\underbrace{p(n-1|n-2;...;1)}_{p(n-1|n-2)}\,p(n-2;...;1)} = ... = p(n|n-1)...p(2|1)p(1) \tag{11}$$

2. Introducing a dummy variable which is *marginalised out* gives

$$p(3;1) = p(3|1)p(1) = \int \mathrm{d}x_2 p(3;2;1) = \int \mathrm{d}x_2 p(3|2)p(2|1)p(1). \tag{12}$$

Dividing by $p(1)$, we find the Chapman-Kolmogorov equation,

$$p(3|1) = \int \mathrm{d}x_2 p(3|2)p(2|1), \tag{13}$$

as given.

3. Substituting the given distribution, one finds

$$I \equiv \int \mathrm{d}x_2 p(3|2)p(2|1) = \int \mathrm{d}x_2 \frac{1}{\sqrt{(4\pi)^2(t_3-t_2)(t_2-t_1)}} \exp\left\{-\left(\frac{(x_3-x_2)^2}{4(t_3-t_2)} + \frac{(x_2-x_1)^2}{4(t_2-1)}\right)\right\} \tag{14}$$

Reducing the argument of exp to a common denominator and completing the squares gives an exponent of

$$\left(\frac{(x_3-x_2)^2}{4(t_3-t_2)} + \frac{(x_2-x_1)^2}{4(t_2-t_1)}\right) \tag{15}$$

$$= -\frac{(t_3-t_1)}{4(t_3-t_2)(t_2-t_1)}\left[x_2 - \frac{x_3(t_2-t_1)+x_1(t_3-t_2)}{(t_3-t_1)}\right]^2 - \frac{(x_3-x_1)^2}{4(t_3-t_1)} \tag{16}$$

The x_2 integral is just a Gaussian integral, yielding

$$\frac{1}{\sqrt{4\pi(t_3-t_1)}} \exp\left\{-\frac{(t_2-t_1)(t_3-t_2)(x_3-x_1)^2}{4(t_2-t_1)(t_3-t_2)(t_3-t_1)}\right\} = p(3|1). \tag{17}$$

4. Integrating over x using $\int \mathrm{d}x p(x,t+\Delta t|x'',t) = 1$ gives

$$a(x'',t) = \int \mathrm{d}x W(x,x'',t). \tag{18}$$

5. The Chapman-Kolmogorov equation gives

$$
\begin{aligned}
p(x,t+\Delta t|x',t') &= \int \mathrm{d}x'' p(x,t+\Delta t|x'',t)p(x'',t|x',t') && (19)\\
&= (1-a(x,t)\Delta t)p(x,t|x',t') + \Delta t \int \mathrm{d}x'' W(x,x'',t)p(x'',t|x',t') + O(\Delta t^2) && (20)\\
&= p(x,t|x',t') + \Delta t \int \mathrm{d}x''(W(x,x'',t)p(x'',t|x',t') - W(x'',x,t)p(x,t|x',t')) + O(\Delta t^2) && (21)
\end{aligned}
$$

Subtracting $p(x,t|x',t')$, dividing through by Δt and letting $\Delta t \to 0$ gives the given master equation.

6. A Taylor expansion in the first argument about x gives

$$
\begin{aligned}
w(x-r,r,t)p(x-r,t) &= w(x,r,t)p(x,t) - r\partial_x w(x,r,t)p(x,t) + \frac{r^2}{2!}\partial_x^2 w(x,r,t)p(x,t) + \cdots && (22)\\
&= \sum_{n=0}^{\infty} \frac{r^n}{n!}(-\partial_x)^n [w(x,r,t)p(x,t)]. && (23)
\end{aligned}
$$

Plugging this into the master equation gives

$$
\partial_t p(x,t) = \int \mathrm{d}r \left\{ \sum_{n=0}^{\infty} \frac{r^n}{n!}(-\partial_x)^n [w(x,r,t)p(x,t)] - w(x,-r,t)p(x,t) \right\}. \tag{24}
$$

Letting $r \to -r$ in the second term (while paying attention to the limits of integration) yields,

$$
\begin{aligned}
\int_{-\infty}^{\infty} -w(x,-r,t)p(x,t)dr \to & \int_{+\infty}^{-\infty} -w(x,r,t)p(x,t)(-dr)\\
&= \int_{+\infty}^{-\infty} w(x,r,t)p(x,t)dr\\
&= \int_{-\infty}^{+\infty} -w(x,r,t)p(x,t)dr
\end{aligned}
$$

which cancels the $n=0$ term. We thus get

$$
\partial_t p(x,t) = \sum_{n=1}^{\infty} (-\partial_x)^n [D_n p], \tag{25}
$$

with $D_n = \frac{1}{n!}\int \mathrm{d}r w(x,r,t) r^n$.

7. $J(\cdot) = -D_1(x,t)(\cdot) + \partial_x[D_2(x,t)(\cdot)]$.

8. Multiply the Fokker-Planck equation by x and integrate over x to find

$$
\partial_t \langle x \rangle = \int x\partial_t p(x,t)dx = \int [-x\partial_x(D_1 p) + x\partial_x^2(D_2 p)]\mathrm{d}x \overset{IBP}{=} \langle D_1 \rangle - D_2 p|_{-\infty}^{\infty} = \langle D_1 \rangle. \tag{26}
$$

9. We will assume that $H(t)$ exists. This assumption is violated for instance if one of the PDFs p_1, p_2 is a delta function. Using $\log x \le x - 1 \forall x \ge 0$, we have,

$$
\begin{aligned}
-\int p_1 \log \frac{p_2}{p_1} dx &\ge \int p_1 \left(\frac{p_2}{p_1} - 1 \right) dx && (27)\\
&= \int p_2 dx - \int p_1 dx = 0 && (28)\\
&\Rightarrow H(t) \ge 0 && (29)
\end{aligned}
$$

Now let's derive the behaviour of $\dot{H}$, introducing $R = \frac{p_1}{p_2}$:

$$
\begin{aligned}
\dot{H}(t) &= \int \dot{p}_1 \log \frac{p_1}{p_2} + \frac{p_1 p_2}{p_1}\left(\frac{\dot{p}_1}{p_2} - \frac{p_1}{p_2^2}\dot{p}_2 \right) dx && (30)\\
&= \int \dot{p}_1 \log R dx - \int R\dot{p}_2 dx, && (31)
\end{aligned}
$$

where the integral in $\dot{p}_1$ vanishes because $p_1(\pm\infty) = 0$. Introducing the Fokker-planck differential operator and its adjoint, $\mathcal{L}(\cdot) = -\partial_x(D_1(\cdot)) + \partial_x^2(D_2(\cdot))$ and $\mathcal{L}^+(\cdot) = [D_1\partial_x + D_2\partial_x^2](\cdot)$ we find

$$
\begin{aligned}
\dot{H}(t) &= \int \left[\dot{p}_1 \ln(R) - \dot{p}_2 \frac{p_1}{p_2} \right] \mathrm{d}x = \int [(\mathcal{L}p_1)\ln(R) - R\dot{p}_2]\,\mathrm{d}x && (32)\\
&= \int [p_1(\mathcal{L}^+ \ln(R)) - R\dot{p}_2]\mathrm{d}x. && (33)
\end{aligned}
$$

Now $\mathcal{L}^+\ln(R) = (D_1 + D_2\partial_x)[R^{-1}\partial_x R] = R^{-1}D_1\partial_x R - D_2(\partial_x \log R)^2 + R^{-1}D_2\partial_x^2 R = R^{-1}\mathcal{L}^+R - D_2(\partial_x \log R)^2$ and thus

$$\begin{aligned}
\dot{H} &= \int p_1 R^{-1}\mathcal{L}^+R - R\dot{p}_2 - p_1 D_2(\partial_x \ln R)^2 \mathrm{d}x && (34)\\
&= \int p_2\mathcal{L}^+R - R\dot{p}_2 - p_1 D_2(\partial_x \ln R)^2 \mathrm{d}x && (35)\\
&= \int R\mathcal{L}p_2 - R\dot{p}_2 - p_1 D_2(\partial_x \ln R)^2 \mathrm{d}x && (36)\\
&= \int R(\dot{p}_2 - \dot{p}_2) - p_1 D_2(\partial_x \ln R)^2 \mathrm{d}x && (37)\\
&= -\int p_1 D_2(\partial_x \ln R)^2 \mathrm{d}x \leq 0. && (38)
\end{aligned}$$

Therefore, as long as $\partial_x \ln R \neq 0$, $H(t)$ decreases. However, it cannot decrease indefinitely, since $H(t) \geq 0$. Thus, as $t \to \infty$, $\partial_x \ln(R) \to 0$, i.e. $R \to const. \equiv 1$ by normalisation of p_i. Hence $p_1 \equiv p_2$ in the limit $t \to \infty$ and the same is true for for any other PDF p_3, meaning that all solutions of the Fokker-Planck equation coincides after long time, no matter from which initial conditions one starts, for general coefficients, as long as $H(t)$ exists, which is often the case.